Online Appendix: Strategy-proof Pareto-improvement

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1 Pareto-constrained participation-maximality as a necessary condition for strategy-proofness-constrained Pareto-efficiency

We demonstrate here that Pareto-constrained participation-maximality, while sufficient, is not a necessary condition for an individually rational and strategy-proof mechanism to be strategy-proofness-constrained Pareto-efficient.

Consider an object allocation problem with $N \equiv \{i_1, i_2, i_3, \dots\}$ and $O \equiv \{a, b, c, \dots\}$. Suppose that T is a singleton, $|N| \geq 3$, $|O| \geq 3$, for each $o \in O$, $\mathcal{F}_o = \{\{i\} : i \in N\} \cup \{\emptyset\}$, $\mathcal{F}$ is Cartesian, and the preferences are strict.

Consider the benchmark mechanism, $\underline{\varphi}$, defined by setting, for each $P \in \mathcal{P}$,

$$\underline{\varphi}_{i_1}(P) = P_{i_1}\text{-max}(O \setminus \{a\}).^{1}$$

$$\underline{\varphi}_{i_2}(P) = \begin{cases} P_{i_2}\text{-max}(O \setminus \underline{\varphi}_{i_1}(P)) & \text{if } \underline{\varphi}_{i_1}(P) \neq c \text{ or} \\ P_{i_2}\text{-max}(O \setminus (\underline{\varphi}_{i_1}(P) \cup \underline{\varphi}_{i_3}(P))) & \text{otherwise} \end{cases}$$

$$\underline{\varphi}_{i_3}(P) = \begin{cases} P_{i_3}\text{-max}(O \setminus (\underline{\varphi}_{i_1}(P) \cup \underline{\varphi}_{i_2}(P))) & \text{if } \underline{\varphi}_{i_1}(P) = c \text{ or} \\ P_{i_3}\text{-max}(O \setminus \underline{\varphi}_{i_1}(P)) & \text{otherwise} \end{cases}$$

and for $k > 3$,

$$\underline{\varphi}_{i_k}(P) = P_{i_k}\text{-max}(O \setminus (\underline{\varphi}_{i_1}(P) \cup \cdots \cup \underline{\varphi}_{i_{k-1}}(P)))$$

In words, this mechanism assigns to i_1 his most preferred object except for a. The remaining objects are distributed among the remaining agents sequentially in the order i_2, i_3, i_4,... if i_1 is not assigned c. The places of i_2 and i_3 are swapped if i_1 is assigned c. Since

[1] Given $P_i \in \mathcal{P}_i$ and $A \subseteq O$, we denote the best element of A according to P_i by P_i-max(A).

i_1 is barred from receiving a, this mechanism is not Pareto-constrained participation-maximal: at each $P \in \mathcal{P}$ such that, for each $i \in N \setminus \{i_1\}, \emptyset\ P_i\ a$ and, for each $o \in O \setminus \{a\}$, $a\ P_{i_1}\ \emptyset\ P_{i_1}\ o$, $\underline{\varphi}_{i_1}(P) = \emptyset$ and a is not assigned to anyone, so $\underline{\varphi}$ is not Pareto-constrained participation-maximal.

While it may be possible to find a strategy-proof mechanism that Pareto-improves $\underline{\varphi}$, we show that no such mechanism is Pareto-constrained participation-maximal. Thus, there *is* a strategy-proof mechanism that cannot be Pareto-improved by another strategy-proof mechanism but is not Pareto-constrained participation-maximal.

To prove this claim, suppose that φ is Pareto-constrained participation-maximal and Pareto-improves $\underline{\varphi}$. Consider $P \in \mathcal{P}$ as follows:

P_{i_1}	P_{i_2}	P_{i_3}	and for $k > 3$,	P_{i_k}
a	b	b		$\emptyset$
$\emptyset$	a	$\emptyset$		
	$\vdots$			

By definition of $\underline{\varphi}$, we have $\underline{\varphi}_{i_2}(P) = b$, and for each $i \in N \setminus \{i_2\}$, $\varphi_i(P) = \emptyset$. Since φ Pareto-improves $\underline{\varphi}$ and φ is Pareto-constrained participation-maximal, $\varphi_{i_1}(P) = a$, $\varphi_{i_2}(P) = b$, and, for each $i \in N \setminus \{i_1, i_2\}$, $\varphi_i(P) = \emptyset$.

Now consider $P'_{i_1} \in \mathcal{P}_{i_1}$ as follows:

P'_{i_1}
a
c
$\vdots$

By definition of $\underline{\varphi}$, we have $\underline{\varphi}_{i_1}(P'_{i_1}, P_{-i_1}) = c$, $\underline{\varphi}_{i_2}(P'_{i_1}, P_{-i_1}) = a$, $\underline{\varphi}_{i_3}(P'_{i_1}, P_{-i_1}) = b$, and for each $i \in N \setminus \{i_1, i_2, i_3\}$, $\underline{\varphi}_i(P'_{i_1}, P_{-i_1})) = \emptyset$. Since this allocation is Pareto-efficient at (P'_{i_1}, P_{-i_1}) and φ Pareto-improves $\underline{\varphi}$, $\varphi(P'_{i_1}, P_{-i_1}) = \underline{\varphi}(P'_{i_1}, P_{-i_1})$. But then $\varphi_{i_1}(P_{i_1}, P_{-i_1}) = a\ P'_{i_1}\ c = \varphi_{i_1}(P'_{i_1}, P_{-i_1})$, so φ is not strategy-proof.

We conclude that no strategy-proof mechanism that Pareto-improves $\underline{\varphi}$ is Pareto-constrained participation-maximal.

Actually, $\underline{\varphi}$ in the above example is *group strategy-proof*.[2] Since no Pareto-constrained participation-maximal strategy-proof mechanism Pareto-improves it, there is a *group* strategy-proof mechanism on the strategy-proofness-constrained Pareto frontier that is not Pareto-constrained participation-maximal.

While $\underline{\varphi}$ does not satisfy Pareto-constrained participation-maximality, it does satisfy a range-based non-wastefulness condition: there is no allocation in its range that Pareto-improves on the chosen allocation in a way that assigns an object to more agents. However, this is a very weak property—even the constant mechanism that always selects $\emptyset$

[2] A mechanism, φ, is **group strategy-proof** if no group of agents can misreport their preferences in a way that at least one member is better off while each member is at least as well off. That is, for each $R \in \mathcal{R}$ and each $S \subseteq N$, there is no $R'_S \in \times_{i \in S} \mathcal{R}_i$, such that for each $i \in S$, $\varphi_i(R'_S, R_{-S})\ R_i\ \varphi_i(R)$ and for some $i \in S$, $\varphi_i(R'_S, R_{-S})\ P_i\ \varphi_i(R)$.

satisfies it—that does not guarantee that a mechanism is strategy-proofness-constrained Pareto-efficient.

2 Stability, non-wastefulness, and Pareto-constrained participation-maximality

We first show that non-wastefulness implies Pareto-constrained participation-maximality.

If $\mu \in \mathcal{F}$ is non-wasteful but not Pareto-constrained participation-maximal, then there is $\nu \in \mathcal{F}$ such that $N(\nu) \supsetneq N(\mu)$ and for each $i \in N$, $\nu(i)\ R_i\ \mu(i)$. By no indifference with the outside option, for each $i \in N(\nu) \setminus N(\mu)$, $\nu(i)\ P_i\ \mu(i) = \emptyset$. Finally, since $|N(\nu)| > |N(\mu)|$, there is $o \in O$ such that $|\nu(o)| > |\mu(o)|$. Thus, μ is wasteful.

We next show that the converse need not hold.

Let T be a singleton, $O = \{o_1, o_2\}, N = \{i_1\}$, $\mathcal{F}_{o_1} = \mathcal{F}_{o_2} = \{\emptyset, \{i_1\}\}$, and $\mathcal{F}$ be Cartesian. Consider $R \in \mathcal{R}$ such that $o_1\ R_{i_1}\ o_2\ R_{i_1}\ \emptyset$. Let $\mu \in \mathcal{F}$ be such that $\mu(i_1) = o_2$. Since $N(\mu) = N$, it is Pareto-constrained participation-maximal. However, consider $\nu \in \mathcal{F}$ such that $\nu(i_1) = o_1$. Since $\nu(i_1)\ P_{i_1}\ \mu(i_1)$ and $1 = |\nu(o_1)| > |\mu(o_1)| = 0$, μ is wasteful.

We now provide an example showing that without any assumptions on choice correspondences, even if they are single-valued, stability may not imply non-wastefulness.

Example 1. *There may be a stable allocation that is not Pareto-constrained participation-maximal.*

Let $N \equiv \{i_1, i_2\}$, $T \equiv \{t_1, t_2\}$, $O \equiv \{o\}$, and $X(o) \equiv \{(i_1, o, t_1), (i_1, o, t_2), (i_2, o, t_1)\}$. Let C_o be such that for each $Y \subseteq X(o)$, if $(i_1, o, t_1) \in Y$ then $C_o(Y) = \{(i_1, o, t_1)\}$ and otherwise $C_o(Y) = \{Y\}$. Let $R \in \mathcal{R}$ be such that $(i_1, o, t_2)\ P_{i_1}\ (i_1, o, t_1)\ P_{i_1}\ \emptyset$ while $(i_2, o, t_1)\ P_{i_2}\ \emptyset$. Let $\mu \equiv \{(i_1, o, t_1)\} \in \mathcal{F}$. Then μ is stable at R. However, it is not Pareto-constrained participation-maximal as there is $\nu \equiv \{(i_1, o, t_2), (i_2, o, t_1)\} \in \mathcal{F}$, which makes every agent better off and $i_2 \notin N(\nu) \setminus N(\mu)$.

Notice that C_o in Example 1 is not size monotonic, though it satisfies IRC and substitutability.

3 School Choice: equivalence of stability and individual rationality, non-wastefulness, and respect of priorities

Let μ be stable with respect to C. By definition of stability it is individually rational. Since C is size monotonic and idempotent, since stability implies non-wastefulness, μ is non-wasteful. If it violates priorities, there are a pair $i, j \in N$ and $o \in O$ such that $\mu(i) = o$, $o\ P_j\ \mu(j)$, and $j \succ_o i$. Since μ is non-wasteful, $|\mu(o)| = q_o$. Since $j \succ_o i$, $C_o(\mu(o) \cup \{j\}) = \{(\mu(o) \setminus \{i\}) \cup \{j\}\}$. This contradicts the stability of μ.

Suppose that μ is individually rational, non-wasteful, and respects priorities. If it is not stable, then there are $o \in O$ and $Y \subseteq N \setminus \mu(o)$ such that for each $i \in Y$, $o\ P_i\ \mu(i)$ and (1) $Y \subseteq Z$ for some $Z \in C_o(\mu(o) \cup Y)$ and (2) $\mu(o) \notin C_o(\mu(o) \cup Y)$. Since, for each $i \in Y$,

$o\ P_i\ \mu(i)$, and μ is non-wasteful, $|\mu(o)| = q_o$. Thus, for each $Z \in C_o(\mu(o) \cup Y)$, $|Z| = |\mu(o)|$. Since $\mu(o) \notin C_o(\mu(o) \cup Y)$, there are $i \in Y$ and $j \in \mu(o)$ such that $i \succ_o j$. This contradicts the assumption that μ respects priorities.

4 Multiple strategy-proof and stable mechanisms without IRC

We provide an example of a capacity-based setting where the ranking according to which agents are chosen depends upon the agents being compared.

Consider a situation where there are two positions for teachers at one school o. There are four candidates $N \equiv \{m_1, m_2, p_1, p_2\}$. There is only one term each teacher can be hired under, so T is a singleton. Let C_o be a single-valued choice correspondence described by the following process. Two of the teachers, m_1 and m_2, specialize in math and the other two, p_1 and p_2, specialize in physics. The math teachers are able to teach physics but not as well as the physics teachers and vice versa. As overall teachers, m_2 is the best, followed by p_1, m_1, and p_2, in that order. If more math specialists are being considered than physics specialists, then the math faculty are more likely to weigh in, so the positions are filled according to how good the candidates are as math teachers. Vice versa if there are more physics specialists. If there are equal numbers of math and physics specialists, the candidates are compared based on their overall teaching ability.

Below, the boxed elements show the choices from each set of candidates.

$$\{m_1, \boxed{m_2, p_1}, p_2\}$$

$$\{\boxed{m_1, m_2}, p_1\} \quad \{\boxed{m_1, m_2}, p_2\} \quad \{m_1, \boxed{p_1, p_2}\} \quad \{m_2, \boxed{p_1, p_2}\}$$

$$\{\boxed{m_1, m_2}\} \quad \{\boxed{m_1, p_1}\} \quad \{\boxed{m_2, p_1}\} \quad \{\boxed{m_1, p_2}\} \quad \{\boxed{m_2, p_2}\} \quad \{\boxed{p_1, p_2}\}$$

$$\{\boxed{m_1}\} \quad \{\boxed{m_2}\} \quad \{\boxed{p_1}\} \quad \{\boxed{p_2}\}$$

Though C satisfies our assumptions of size monotonicity and idempotence, it violates IRC.

For each $S \subseteq N$ such that $|S| \leq 2$, let $\mu^S \in \mathcal{F}$ be such that it assigns agents in S to o and leaves the others unassigned. That is, $\mu^S(o) = S$ and for each $i \in N \setminus S, \mu^S(i) = \varnothing$. For each $P \in \mathcal{P}$, let $G(P) \equiv \{i \in N : o\ P_i\ \varnothing\}$.

Consider the mechanism φ defined by setting, for each $P \in \mathcal{P}$,

$$\varphi(P) \equiv \begin{cases} \mu^{\{m_1, m_2\}} & \text{if } \{m_1, m_2\} \subseteq G(P), \\ \mu^{\{p_1, p_2\}} & \text{if } \{p_1, p_2\} \subseteq G(P) \text{ and } \{m_1, m_2\} \not\subseteq G(P), \\ \mu^{G(P)} & \text{otherwise.} \end{cases}$$

Claim 1. *φ is strategy-proof and stable.*

Proof. We first establish that φ is stable by considering four cases.

Case 1: $m_1, m_2 \in G(P)$. Then $\varphi(P) = \mu^{\{m_1,m_2\}}$. Regardless of whether $p_1, p_2 \in G(P)$, there is no $Y \subseteq G(P) \setminus \{m_1, m_2\}$ such that $Y \subseteq C_o(\mu^{\{m_1,m_2\}} \cup Y)$. Thus, $\varphi(P)$ is stable.
Case 2: $m_1 \notin G(P)$ but $m_2 \in G(P)$. If $p_1, p_2 \in G(P)$, then $\varphi(P) = \mu^{\{p_1,p_2\}}$. Since $C_o(\{m_2, p_1, p_2\}) = \{p_1, p_2\}$, $\varphi(P)$ is stable. Otherwise, $\varphi(P) = \mu^{G(P)}$ and each agent receives his top choice. Thus $\varphi(P)$ is stable.
Case 3: $m_1 \in G(P)$ but $m_2 \notin G(P)$. This is symmetric to Case 2.
Case 4: $m_1, m_2 \notin G(P)$. Since, $\varphi(P) = \mu^{G(P)}$ and each agent receives his top choice, $\varphi(P)$ is stable.

To show that φ is strategy-proof, we again consider the same four cases.
Case 1: $m_1, m_2 \in G(P)$. Then $\varphi(P) = \mu^{\{m_1,m_2\}}$ and neither m_1 nor m_2 can benefit by misreporting his preferences. Regardless $P_{\{p_1,p_2\}}$, φ selects $\mu^{\{m_1,m_2\}}$ so neither of p_1 or p_2 can benefit by misreporting his preference either.
Case 2: $m_1 \notin G(P)$ but $m_2 \in G(P)$. If $p_1, p_2 \in G(P)$, then $\varphi(P) = \mu^{\{p_1,p_2\}}$, so neither p_1 nor p_2 benefits by misreporting and since φ selects $\mu^{\{p_1,p_2\}}$ regardless of m_2's preference, he has no incentive to misreport either. Otherwise, $\varphi(P) = \mu^{G(P)}$ and no agent can benefit by misreporting since he receives his top choice.
Case 3: $m_1 \in G(P)$ but $m_2 \notin G(P)$. This is symmetric to Case 2.
Case 4: $m_1, m_2 \notin G(P)$. Since $\varphi(P) = \mu^{G(P)}$, no agent can benefit by misreporting since he receives his top choice. □

Now, consider the mechanism φ' defined by setting, for each $P \in \mathcal{P}$,

$$\varphi'(P) \equiv \begin{cases} \mu^{\{p_1,p_2\}} & \text{if } \{p_1,p_2\} \subseteq G(P), \\ \mu^{\{m_1,m_2\}} & \text{if } \{m_1,m_2\} \subseteq G(P) \text{ and } \{p_1,p_2\} \not\subseteq G(P), \\ \mu^{G(P)} & \text{otherwise.} \end{cases}$$

Since it is symmetric to φ, φ' is also strategy-proof and stable. In fact, both of these mechanisms are group strategy-proof. Neither of these mechanisms is generated by a cumulative offer algorithm (Hatfield and Milgrom, 2005), which, regardless of the order, outputs the unstable allocation $\mu^{\{m_2,p_1\}}$ for each $P \in \mathcal{P}$ such that $G(P) = N$.

5 On non-wastefulness

Extending non-wastefulness We present two examples that illustrate challenges to extending the non-wastefulness definition of Balinski and Sönmez (1999) beyond the school choice model, even within object allocation.

Example 2. *Feasibility not capacity-based.*

Let T be a singleton, $O \equiv \{o\}$, $N \equiv \{i_1, i_2, i_3\}$, and $\mathcal{F}_o \equiv \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_2, i_3\}\}$. Consider $P \in \mathcal{P}$ as follows:

P_{i_1}	P_{i_2}	P_{i_3}
o	o	o
$\emptyset$	$\emptyset$	$\emptyset$

What is the "capacity" of o? The largest set of agents that may consume o contains two elements while the smallest non-trivial set contains only one. If we naïvely extend non-wastefulness by setting the capacity of o to be two, then allocating it to i_1 would be wasteful, even though this is the only allocation where i_1 receives his top choice. On the other hand if we set the capacity of o to be one, then allocating it to i_2 alone would *not* be wasteful even though o could be assigned to i_3 as well. Neither of these is sensible. Thus, non-wastefulness cannot be extended in a way that relies on a fixed capacity for each object.

Example 3. *Complementarities in feasibility.*

Let T be a singleton, $O \equiv \{o_1, o_2\}$, $N \equiv \{i_1, i_2\}$, $\mathcal{F}_{o_1} \equiv \{\emptyset, \{i_1, i_2\}\}$, $\mathcal{F}_{o_2} \equiv \{\emptyset, \{i_1\}, \{i_2\}\}$, and $\mathcal{F}$ be Cartesian (though not capacity-based). Consider $P \in \mathcal{P}$ as follows:

P_{i_1}	P_{i_2}
o_1	o_2
o_2	o_1
$\emptyset$	$\emptyset$

There are two allocations of interest. The first assigns o_1 to both agents. This is the only allocation where i_1 receives his top choice. The second assigns $\emptyset$ to i_1 and o_2 to i_2. This is the only allocation where i_2 receives his top choice. At either of these allocations there is an agent who prefers the unallocated object to what he receives. However, the only way he can be assigned this unallocated object is by making the other agent worse off. A sensible definition of non-wastefulness, in the general setting, should not rule either of these allocations out.

Though there is no fixed notion of capacity, as demonstrated by Example 2, non-wastefulness should seek to ensure that each object is utilized to the greatest extent possible. Yet, as demonstrated by Example 3, it should take care to ensure that increasing the utilization of an object by allocating it to agents who prefer it does not harm other agents.

Our definition of wastefulness summarizes the discussion above. Given $P \in \mathcal{P}$, $\mu \in \mathcal{F}$ is wasteful if there are $o \in O$, $i \in N$, and $\nu \in \mathcal{F}$, such that *(1)* $|\nu(o)| > |\mu(o)|$, so that ν allocates o to more agents than μ does, *(2)* $\nu(i)\ P_i\ \mu(i)$, so that i prefers his assignment at ν to that at μ, and *(3)* for each $j \in N \setminus \{i\}$, $\nu(j)\ R_i\ \mu(j)$, so that no agent is worse off at ν compared to μ. As we now demonstrate, this definition is an extension of the definition of that by Balinski and Sönmez (1999) to our more general setting.

Claim 2. *Suppose that T is a singleton and $\mathcal{F}$ is capacity-based. An allocation μ is non-wasteful by the definition of Balinski and Sönmez (1999) if and only if it is non-wasteful.*

Proof. Suppose that μ is wasteful by the definition of Balinski and Sönmez (1999). Then there are $o \in O$ and $i \in N$ such that $o\ P_i\ \mu(i)$ and $|\mu(i)| < q_o$. Let $\nu \equiv (\mu \cup \{(i, o)\}) \setminus \mu(i)$. Then $|\nu(o)| = |\mu(o)| + 1 \leq q_o$ and, for each $o' \in O \setminus \{o\}$, $|\mu(o')| - 1 \leq |\nu(o')| \leq |\mu(o')| \leq q_{o'}$. Thus, $\nu \in \mathcal{F}$ and $|\nu(o)| > |\mu(o)|$. Furthermore, $\nu(i)\ P_i\ \mu(i)$ while for each $j \in N \setminus \{i\}, \nu(i) = \mu(i)$. Thus, μ is wasteful.

Suppose that μ is non-wasteful by the definition of Balinski and Sönmez (1999). Consider $\nu \in \mathcal{F}$ such that, for each $i \in N$, $\nu(i)\ R_i\ \mu(i)$ and for some $i \in N$, $\nu(i)\ P_i\ \mu(i)$. Let $o \in O$.

If $|\mu(o)| < q_o$, by the Balinski and Sönmez definition of non-wastefulness, there is no $i \in N$ such that $o\ P_i\ \mu(i)$. So $|\nu(o)| \leq |\mu(o)|$. If $|\mu(o)| = q_o$, by feasibility of ν, $|\nu(o)| \leq q_o = |\mu(o)|$. Thus, μ is non-wasteful. □

6 Impossibility result for a simple excludable public goods model

If the highest threshold for an agent in N_L is strictly lower than the lowest threshold for an agent in N_R, then no compromise is possible since there are no locations that are acceptable to at least one member of each group.

Label as $l_1, \ldots, l_k$ the agents in N_L and as $r_1, \ldots, r_{n-k}$ the agents in N_R.

Suppose that φ is strategy-proof, individually rational, and Pareto-efficient. Let t be a profile of thresholds such that no compromise is possible. Pareto-efficiency and individual rationality require that $\varphi(t)$ is either $(0, N_L)$ or $(1, N_R)$. Suppose that it is $(0, N_L)$ (the argument is symmetric if it is $(1, N_R)$).

Step 1: *For each t' such that no compromise is possible, $\varphi(t') = (0, N_L)$.*

Let

$$\underline{\alpha} \equiv \min_{i \in N_L} \min\{t_i, t'_i\}$$

For each $i \in N_L$, let $\underline{t}_i \in (0, \underline{\alpha})$. Since $\varphi(t) = (0, N_L)$, by strategy-proofness, Pareto-efficiency and individual rationality, $\varphi(\underline{t}_{l_1}, t_{-l_1}) = (0, N_L)$. Otherwise, if $\varphi(\underline{t}_{l_1}, t_{-l_1}) = (x, S)$, if $x \neq 0$ or $l_1 \notin S$, then l_1 benefits by reporting t_{l_1} rather than $\underline{t}_{l_1}$. Furthermore, by Pareto-efficiency, $N_L \subseteq S$ and by individual rationality, since 0 is below the threshold of each member of N_R, $S = N_L$. Repeating this argument, we replace, for each member of N_L, t_i by $\underline{t}_i$ to conclude that $\varphi(\underline{t}_{N_L}, t_{N_R}) = (0, N_L)$. By a similar argument, we replace, one at a time for each $i \in N_R$, t_i by t'_i to see that $\varphi(\underline{t}_{N_L}, t'_{N_R}) = (0, N_L)$. Then, we use the same argument for a third time to replace, for each $i \in N_L$, $\underline{t}_i$ by t'_i to deduce that $\varphi(t') = (0, N_L)$.

Step 2: *For each t', $\varphi(t') = (0, S)$ such that $N_L \subseteq S$.*

Let t' be a profile of thresholds and let $\underline{\beta} = \min_{i \in N_R} t'_i$. For each $\tilde{t}_{N_L}$, let $N_{\underline{\beta}}(\tilde{t}_{N_L}) = \{i \in N_L : \tilde{t}_{N_L} \geq \underline{\beta}\}$. We prove by induction on the cardinality of $N_{\underline{\beta}}(\tilde{t}_{N_L})$ that for each $\tilde{t}_{N_L}$, $\varphi(\tilde{t}_{N_L}, t'_{N_R}) = (0, S)$ such that $N_L \subseteq S$.

Step 1 establishes the base case of $N_{\underline{\beta}}(\tilde{t}_{N_L}) = \varnothing$ since no compromise is possible.

As an induction hypothesis, assume that for each $\tilde{t}_{N_L}$ such that $|N_{\underline{\beta}}(\tilde{t}_{N_L})| < k$, $\varphi(\tilde{t}_{N_L}, t'_{N_R}) = (0, S)$ such that $N_L \subseteq S$.

We now show the induction step that for each $\tilde{t}_{N_L}$ such that $|N_{\underline{\beta}}(\tilde{t}_{N_L})| = k$, $\varphi(\tilde{t}_{N_L}, t'_{N_R}) = (0, S)$ such that $N_L \subseteq S$. Suppose not. Then $\varphi(\tilde{t}_{N_L}, t'_{N_R}) = (x, S)$ for some $S \subseteq N$ and $x \neq 0$. Let $i \in N_{\underline{\beta}}(\tilde{t}_{N_L})$ and $\underline{t}_i \in (0, \underline{\beta})$. Then, $|N_{\underline{\beta}}(\underline{t}_i, \tilde{t}_{N_L \setminus \{i\}})| = k - 1$. By the induction hypothesis, $\varphi(\underline{t}_i, \tilde{t}_{N_L \setminus \{i\}}, t'_{N_R}) = (0, \tilde{S})$ where $i \in N_L \subseteq \tilde{S}$. Since $x \neq 0$, regardless of S,

$$(0, \tilde{S}) = \varphi(\underline{t}_i, \tilde{t}_{N_L \setminus \{i\}}, t'_{N_R})\ \tilde{P}_i\ \varphi(\tilde{t}_i, \tilde{t}_{N_L \setminus \{i\}}, t'_{N_R}) = (x, S).$$

This contradicts the strategy-proofness of φ. Thus, $x = 0$ and by Pareto-efficiency, $N_L \subseteq S$.

References

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