The Impossibility of Strategy-proof, Pareto efficient, and Individually Rational Rules for Fractional Matching*

Samson Alva
University of Texas at San Antonio
samson.alva@gmail.com

Vikram Manjunath
University of Ottawa
vikram@dosamobile.com

June 7, 2019

Abstract

For a model of fractional matching, interpreted as probabilistic matching, together with the allocation of non-negative amounts of money, we show that strategy-proofness, ex post Pareto efficiency of the matching, and a weak version of ex ante individual rationality are incompatible when each agent's utility is a linear function of both their fractional assignment and money. We identify some avenues to escape this impossibility.

JEL classification: C71, C78, D51

Keywords: *fractional, matching, strategy-proofness, Pareto efficiency, individual rationality*

1 Introduction

We study the problem of fractionally matching agents to each other alongside the allotment of a single divisible good (Roth et al., 1993; Manjunath, 2016). Normalizing the total availability of each agent to one, we interpret a fractional matching as a lottery over deterministic matchings. Each agent's preference is quasilinear over fractional matchings and the divisible "money" good. Given the lottery interpretation, we focus on expected utility preferences over fractional matchings. In particular, we assume that each agent seeks to maximizes their expected utility, and that their Bernoulli utility is distinct across potential partners.[1] Each agent, therefore, has a strict preference over potential partners and remaining alone that is invariant to the amount of money they consume. We may make our judgements in regards to properties of allocations either before or after the lottery over matchings is conducted.

*We thank William Thomson for helpful comments. We are also grateful for the useful comments and suggestions by the editor and two anonymous reviewers.

[1] We have chosen to use singular "they/them" as a gender-neutral pronoun in what follows.

In what follows, we use the terms *ex ante* and *ex post* in reference to the stages before and after the resolution of the lottery, rather than before and after agents learn about the state of the world as in the Bayesian mechanism design literature.[2]

The objects of our study are rules that choose allocations as a function of agents' preferences. An allocation consists of two parts: a lottery over deterministic matchings of the agents and a distribution of the available amount of money in the economy. We assume *non-negativity of money consumption*: feasibility requires that no agent consumes a negative amount of money.[3] We show that no strategy-proof[4] rule is ex post Pareto efficient in matching[5] and ex ante individually rational.[6]

For the deterministic marriage (and roommate) problem without money (Gale and Shapley, 1962), no individually rational and Pareto efficient rule is strategy-proof (Alcalde and Barberà, 1994; Sönmez, 1999). Such a problem can be represented as a fractional matching problem where the amount of money to divide is zero, allocations are deterministic, and preferences over partners are strict and have a linear utility representation.[7] Our impossibility result may be seen as strengthening the classic impossibility in two ways. First, we show that the availability of money does not alleviate the impossibility. Second, we drop the requirements that the rule be deterministic and use only ordinal preference information, and considerably weaken individual rationality. While we have assumed that all preferences with strict orderings over partners are available, our proof technique only requires that it be possible for each agent to have two indifference classes among acceptable partners. However, if agents are indifferent between all acceptable partners, that is if preferences are *dichotomous*, then strategy-proofness, *ex ante* Pareto efficiency in matching, and *ex post* individual rationality are indeed compatible (Bogomolnaia and Moulin, 2004).

[2] In the terminology of the Bayesian mechanism design literature, every efficiency and individual rationality concept we define conducts evaluations at what that literature would call the "ex post" stage (Holmström and Myerson, 1983). Indeed, this is the stage at which such evaluations are typically made in the literature on dominant-strategy mechanism design.

[3] Note that we allow free disposal of money. Since not *all* of the available money has to be distributed, we do not impose *budget balance*.

[4] Strategy-proofness means that no agent receives a preferable assignment, from an ex ante perspective, by misreporting their preferences.

[5] Ex post Pareto efficiency in matching means that for each deterministic matching that can be realized, no agent can be made better off without making another agent worse off. Note that this says nothing about how money is allocated.

[6] Ex ante individual rationality means that no agent finds remaining single and consuming no money preferable to the fractional matching and amount of money they are assigned. This is a great deal weaker than *ex post* individual rationality, which requires that no agent finds remaining single with no money preferable to consuming their assigned amount of money with some deterministic matching in the support of chosen fractional matching.

[7] The impossibility holds even for random rules satisfying the ex post versions of Pareto efficiency and individual rationality, the latter of which is the most demanding variant (Gudmundsson, forthcoming).

For the probabilistic allocation of objects without money, strategy-proofness is incompatible with Pareto efficiency and equal treatment of equals (Zhou, 1990).[8] This problem may be encoded in the bipartite version of our model by assuming that agents on one side are indifferent between all partners. However, our impossibility does not hold for this special case: every serial dictatorship is strategy-proof, *ex ante* Pareto efficient, and *ex post* individually rational. Indeed, the set of Pareto efficient and individually rational deterministic matchings expands greatly at the limit of strict preferences for both sides tending towards one side having complete indifference. Thus, results on this limiting domain have little bearing on the model that we study.

The difficulty with obtaining a strategy-proof, Pareto efficient, and individually rational rule is a familiar one in other economic settings. Such rules do not exist for exchange economies with classical preferences over at least two divisible private goods (Hurwicz, 1972; Serizawa, 2002; Momi, 2017). Even for exchange economies with one or more public goods produced linearly from one private good and for social choice with lotteries over alternatives, strategy-proofness and Pareto efficiency lead to dictatorial rules (Schummer, 1999). On the surface, these results may appear very close to ours: the space of lotteries over social alternatives is a simplex, as is the space of fractional matchings, and preferences have linear utility representations. Also, a fractional matching between two agents bears resemblance to a public good, its consumption being non-rival between the pair. However, our results differ in important ways: (1) We appeal to ex post Pareto efficiency of only the matching rather than full ex ante Pareto efficiency of the entire allocation, allowing for money to be "disposed of." (2) The feasible set in our model is the product of two simplices, a fractional matching and a division of the available money, and therefore not itself a simplex. (3) We restrict attention to the set of linear utilities such that each agent is indifferent between two corners of the matching simplex if and only if they assign them the same mate, as opposed to the set of all possible linear utilities. (4) Interpreting fractional matching between a pair as a public good, each agent's availability would have to be a private good that has no value to any other agent, except as an input into a Leontieff, as opposed to linear, technology that produces the matching. So, to the best of our knowledge, none of the existing impossibility results regarding strategy-proof and efficient rules for economies with private or public goods covers ours.

Our result is also related to the well-known incompatibility of strategy-proofness with

[8] With comparisons based on stochastic dominance, the corresponding notions of efficiency, fairness, and strategy-proofness are incompatible (Bogomolnaia and Moulin, 2001; Nesterov, 2017) even if the domain of ordinal preferences is restricted (Kasajima, 2013; Chang and Chun, 2016).

allocative efficiency[9] and budget balance[10] in the transferable utility setting, where the consumption of money is not restricted by a lower bound (Green and Laffont, 1977; Holmström, 1979).[11] This incompatibility holds even for matching with transfers (Yenmez, 2013, 2015). Allocative efficiency is a stronger requirement than (ex ante or ex post) Pareto efficiency in matching under non-negative consumption of money.[12] Consequently, our results are not implied by these impossibilities on the domain of transferable utility problems. This distinction, on the basis of the efficiency criterion, also applies to the related results where strategy-proofness is weakened to Bayesian incentive compatibility.[13]

The remainder of the paper is organized as follows. We define the model and required concepts in Section 2. We present and discuss our impossibility results in Section 3. We consider some ways to obtain possibility results in Section 4. We prove our results in Appendix A.

2 The Model

A *fractional matching model* consists of a finite set of agents N. Each agent i has a set of possible partners $J_i \subseteq N$, with $i \in J_i$. If j is a possible partner of i, then i is a possible partner of j. That is, $j \in J_i$ implies $i \in J_j$. We study both bipartite and non-bipartite matching. In bipartite matching, N is partitioned into two non-empty sets, M and W, such that for each $m \in M$, $J_m = W \cup \{m\}$, and for each $w \in W$, $J_w = M \cup \{w\}$. In non-bipartite matching, for each $i \in N$, $J_i = N$.

Each agent has unit availability. For each $i \in N$, a *fractional assignment* divides their availability between partners in J_i (this includes being alone). That is, a fractional assignment for i is

$$\pi_i \in \Delta_i \equiv \left\{ y_i \in \mathbb{R}_+^{J_i} : \sum_{j \in J_i} y_{ij} = 1 \right\}.$$

For each $j \in J_i \setminus \{i\}$, π_{ij} represents the amount that i is matched to j, while π_{ii} is the amount they remain alone. For each $i \in N$ and each $j \in J_i$, let $\delta_i^j \in \Delta_i$ be such that $\delta_{ij}^j = 1$.

[9] Allocative efficiency means that the fractional matching maximizes the sum of agents' match utilities.

[10] Budget balance means that the total allocation of money equals the amount available in the economy.

[11] For the allocation of objects and a finite amount of money *with* non-negative consumption of money, whether strategy-proofness and Pareto efficiency (that is, allocative efficiency and budget balance) together imply dictatorship remains an open question, as pointed out by Schummer (2000).

[12] Non-negative consumption of money implies that agents are budget constrained, so utility is only boundedly transferable. Therefore, allocative efficiency is not a necessary condition for Pareto efficiency in matching.

[13] See the literature following Myerson and Satterthwaite (1983), particularly on partnership dissolution (Cramton et al., 1987; Yenmez, 2012).

Consider a profile of fractional assignments, $(\pi_i)_{i\in N} \in \Delta \equiv \times_{i\in N}\Delta_i$. For this profile to reflect a matching, for each $i \in N$ and $j \in J_i$, π_{ij}, the amount that i is matched to j, should equal π_{ji}, the amount that j is matched with i. We represent any such $(\pi_i)_{i\in N}$ as a $N \times N$ symmetric bistochastic matrix $\pi \in [0,1]^{N\times N}$. Let Σ be the set of all such π that have integer-valued entries. So, for each $(\pi_i)_{i\in N} \in \Delta$ that satisfies the symmetry condition for a matching and such that for each $i \in N$ and $j \in J_i$, $\pi_{ij} \in \{0,1\}$, the corresponding symmetric bistochastic matrix is in Σ.

A natural interpretation of this model is that of probabilistic matching. We carry out our analysis with this interpretation. Every $\sigma \in \Sigma$ is a *deterministic matching*. In particular, for each $i \in N$ and $j \in J_i$, $\sigma_{ij} = 1$ represents i and j matched with certainty. Let Π be the convex hull of Σ. Each $\pi \in \Pi$ is a *fractional matching*. Such π represents a lottery over deterministic matchings.[14]

Aside from being matched to one another, agents also consume a divisible private good that we call *money*. Let Ω be the total non-negative quantity of this good in the economy. Since i consumes a fractional assignment along with an amount of money, their consumption set is $X_i \equiv \Delta_i \times \mathbb{R}_+$. Let $X \equiv \mathop{\times}\limits_{i\in N} X_i$.

An *allocation* is $(\pi, z) \in X$ such that π is a fractional matching and z is a *feasible* distribution of available money. That is, $\pi \in \Pi$ and $\sum_{i\in N} z_i \leq \Omega$.[15] Let $\mathcal{Z}$ be the set of allocations. Since money consumption is non-negative, when $\Omega = 0$ no monetary transfers are feasible.

For each $i \in N$, i's preference over X_i is represented by a utility function $u_i : X_i \to \mathbb{R}$. Let $\mathcal{U}_i^{lin}$ be the set of all linear utility functions that are increasing in money.[16] Since each $\pi \in \Pi$ is a probability distribution, these utility functions represent von Neumann-Morgenstern expected utility preferences over the matching component of an allocation. For each $u_i \in \mathcal{U}_i^{lin}$, let $R(u_i)$ be the preference relation over i's partners induced by u_i. That is, for each $i \in N$, let $R(u_i)$ be the complete preorder over J_i such that for each pair $j, k \in J_i$, $j\ R(u_i)\ k$ if and only if $u_i(\delta_i^j, 0) \geq u_i(\delta_i^k, 0)$. Let $\mathcal{U}_i^{slin}$ be the set of all linear utility functions for i that induce *strict* preferences over i's partners. That is, for each $u_i \in \mathcal{U}_i^{lin}$, $u_i \in \mathcal{U}_i^{slin}$ if and only if $R(u_i)$ is a linear order over J_i.

[14] Given our probabilistic interpretation, not every $(\pi_i)_{i\in N} \in \Delta$ that satisfies the symmetry condition is necessarily a fractional matching. This is because a symmetric bistochastic matrix may not be equivalent to any convex combination of symmetric permutation matrices. For the bipartite case with N partitioned into M and W, there is a one-to-one correspondence between symmetric profiles in Δ and $M \times W$ bistochastic matrices. So, by the Birkhoff-von Neumann Theorem (Birkhoff, 1946; von Neumann, 1953), every symmetric profile in Δ is in the convex hull of Σ. That is, the set of fractional matchings, Π, corresponds to the set of symmetric profiles in Δ. This equivalence does not hold generally for the non-bipartite case.

[15] This restriction on z is often referred to as "budget feasibility" or "weak budget balance."

[16] For each $i \in N$ and each $u_i \in \mathcal{U}_i^{lin}$, there exists $v_i \in \mathbb{R}^{J_i}$ and $\gamma \in \mathbb{R}_{++}$ such that $u_i(\pi_i, z_i) = v_i \cdot \pi_i + \gamma z_i$.

We show our results for $\mathcal{U}^{slin} \equiv \bigtimes_{i \in N} \mathcal{U}_i^{slin}$, the subdomain of problems with linear utility that are increasing in money *and* induce strict preferences over partners.[17]

Since we fix N, $(J_i)_{i \in N}$, and Ω, a *problem* is fully described by a profile of utility functions $u \in \mathcal{U}^{slin}$. A *rule*, $\varphi : \mathcal{U}^{slin} \to \mathcal{Z}$, assigns to each problem an allocation.

Properties of Allocations and Rules Let $u \in \mathcal{U}^{slin}$ and $(\pi, z) \in \mathcal{Z}$.

If each agent is at least as well off at (π, z), *before* lottery π is conducted, as remaining alone and consuming zero money, we say that (π, z) is *ex ante individually rational at* u. That is, (π, z) is ex ante individually rational if for each $i \in N$, $u_i(\pi_i, z_i) \geq u_i(\delta_i^i, 0)$. This is weaker than the requirement that *after* the lottery π is conducted, the realized allocation satisfies each agent's welfare lower-bound of being unmatched with no money.[18]

We say that a deterministic matching is *Pareto efficient* if there is no deterministic matching that each agent finds at least as desirable and at least one agent prefers. That is, $\sigma \in \Sigma$ is *Pareto efficient* if there is no $\sigma' \in \Sigma$ such that for each $i \in N$, $u_i(\sigma'_i, 0) \geq u_i(\sigma_i, 0)$, and for some $i \in N, u_i(\sigma'_i, 0) > u_i(\sigma_i, 0)$.[19] We say that a fractional matching is *ex post Pareto efficient* if it can be conducted so that a Pareto efficient deterministic matching is realized. That is, $\pi \in \Pi$ is ex post Pareto efficient if and only if it is a convex combination of Pareto efficient deterministic matchings. We say that an allocation $(\pi, z) \in \mathcal{Z}$ is *ex post Pareto efficient in matching* if π is ex post Pareto efficient. This efficiency property is mild in two respects. First, it applies only to the matching dimension of the allocation and makes no statement about how money is allocated. Second, the welfare statement is made *after* the conduct of the lottery as opposed to *before*: the stronger ex ante notion would require that there be no alternative fractional matching that would increase some agent's utility without decreasing another's. Thus, to say that (π, z) is ex post Pareto efficient in matching is clearly weaker than the usual notion of Pareto efficiency: (π, z) is *fully Pareto efficient* if there is no other allocation $(\pi', z') \in \mathcal{Z}$ such that for each $i \in N$, $u_i(\pi', z') \geq u_i(\pi, z)$, and for some $i \in N$, $u_i(\pi', z') > u_i(\pi, z)$.

Finally, let $\varphi : \mathcal{U}^{slin} \to \mathcal{Z}$ be a rule. We say that φ is *strategy-proof* if, for each $u \in \mathcal{U}^{slin}$

[17] Our results hold even if we normalize the marginal utility of money to one and the utility from being matched to oneself to zero.

[18] Suppose that Ω units of money were originally distributed as part of a private endowment $(\omega_i)_{i \in N} \in \mathbb{R}_+^N$ such that $\sum_{i \in N} \omega_i = \Omega$. In such an economy, the natural welfare lower-bound for $i \in N$ would be $u_i(\delta_i^i, \omega_i)$. However, ex ante individual rationality as we have defined it is weaker than satisfying this alternative lower-bound, since ω_i is non-negative and utility is increasing in money.

[19] Because utility is separable in money, the utility comparisons do not depend on the level of money consumption use in the definition.

and each $i \in N$, there is no $u_i' \in \mathcal{U}_i^{slin}$ such that

$$u_i(\varphi_i(u_i', u_{-i})) > u_i(\varphi_i(u)).$$

3 Impossibility

Our main result, described by two theorems, is that no strategy-proof and ex ante individually rational rule always selects an ex post Pareto efficient matching, no matter the available quantity of money Ω. We note here some important points about this result. First, as described above, our proof only appeals to linear utility functions that induce strict preferences over partners. When fractional matchings are interpreted as a division of time across partners (Manjunath, 2016), our result is particularly robust: in the domain of all concave utility functions over fractional matchings, the domain of linear utility functions is negligible. Second, as we have also pointed out above, our individual rationality and efficiency axioms are very weak. In particular, we do not require an efficient allocation of money, but only ex post Pareto efficiency of the matching. When money is available, it then permits the designer some potential latitude in satisfying the constraints imposed by strategy-proofness and ex ante individual rationality, without the burden of having to allocate money in an efficient manner. Despite this, we show below that these axioms are incompatible.

We begin with the bipartite case, with N partitioned into M and W.

Theorem 1. *For a bipartite model, if $|M| \geq 2$ and $|W| \geq 2$, then no rule is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational.*

We illustrate the core argument of the proof in Figures 1 and 2 for the case without money ($\Omega = 0$). When there is money, the need for an additional (fourth) dimension makes a diagrammatic representation infeasible. The proof is in Appendix A.

Next, we consider the non-bipartite model. Since the non-bipartite model places no restriction on the set of possible partners, the set of feasible matchings expands. Unfortunately, as long as there are at least three agents, the impossibility persists.

Theorem 2. *For a non-bipartite model, if $|N| \geq 3$, then no rule is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational.*

To prove Theorem 2, consider first the case where $|N| \geq 4$. Fix a bi-partition of the agents such that there are at least two agents on each "side". The domain of utility profiles in the bipartite model with this fixed partition corresponds to the subdomain of utility profiles in the original non-bipartite model where each agent gets higher utility from being

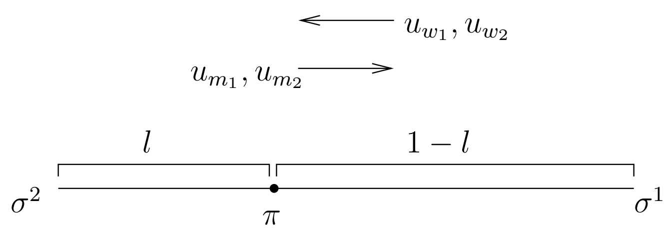
$R(u_{m_1})$	$R(u_{m_2})$	$R(u'_{m_1})$	$R(u'_{m_2})$
w_1	w_2	w_1	w_2
w_2	w_1	m_1	m_2
m_1	m_2	w_2	w_1

$R(u_{w_1})$	$R(u_{w_2})$
m_2	m_1
m_1	m_2
w_1	w_2

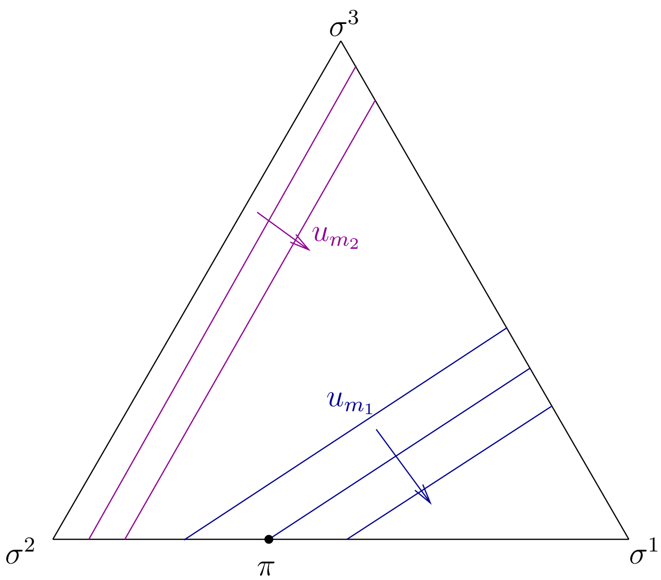
$$\overset{\sigma^1}{\begin{pmatrix} m_1 \leftrightarrow w_1 \\ m_2 \leftrightarrow w_2 \end{pmatrix}} \qquad \overset{\sigma^2}{\begin{pmatrix} m_1 \leftrightarrow w_2 \\ m_2 \leftrightarrow w_1 \end{pmatrix}}$$

$$\overset{\sigma^3}{\begin{pmatrix} m_1 \quad\;\; w_2 \\ m_2 \leftrightarrow w_1 \end{pmatrix}} \qquad \overset{\sigma^4}{\begin{pmatrix} m_1 \leftrightarrow w_2 \\ m_2 \quad\;\; w_1 \end{pmatrix}}$$

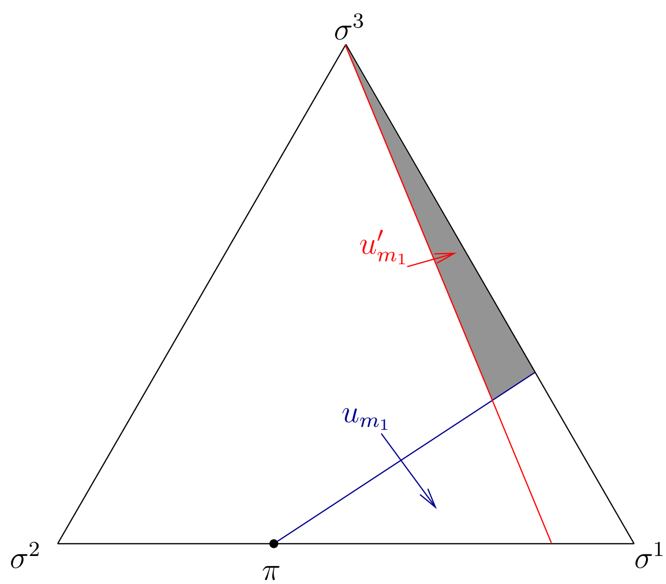
(a) Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$, and $u, u' \in \mathcal{U}^{slin}$ be such that they induce the above preferences over partners and the indifference planes of $u_{m_1}, u'_{m_1}, u_{m_2}$, and u'_{m_2} are as in Figures 1b, 1c, and 2c. Let $\sigma^1, \sigma^2, \sigma^3, \sigma^4 \in \Sigma$ be as indicated.



(b) At u, ex post Pareto efficiency in matching requires $\pi \equiv \varphi(u)$ to be a convex combination of σ^1 and σ^2. The figure indicates directions of increasing preferences for each agent. We consider the case of $l \leq \frac{1}{2}$ as the proof for the alternative case is analogous.

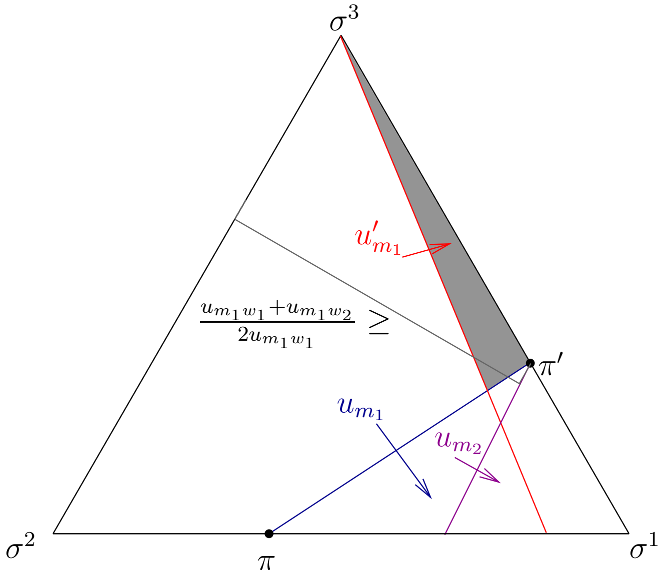


(c) At (u'_{m_1}, u_{-m_1}), ex-post Pareto efficiency in matching requires φ to choose a convex combination of σ^1, σ^2, and σ^3. Indifference lines of u_{m_1} are flatter than the σ^2-σ^3 face of the simplex since m_1 is fully single at σ^3. On the other hand, indifference lines of u_{m_2} are parallel to it since m_2 is matched fully to w_1 by both σ^2 and σ^3.

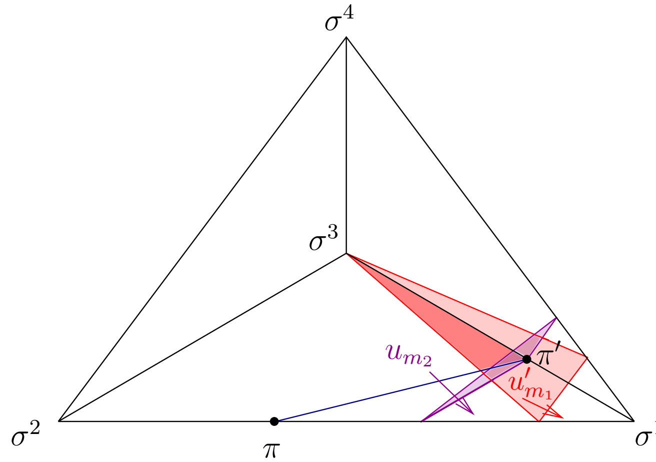


(d) We define u'_{m_1} to be such that π is worse under u'_{m_1} for m_1 than being fully unmatched. Ex ante individual rationality requires that $\varphi(u'_{m_1}, u_{-m_1})$ be to the right of the indifference line of u'_{m_1} through σ^3. Strategy-proofness requires that it be above (per the orientation of this figure) the indifference line of u_{m_1} through π, otherwise at the true preference u_{m_1}, m_1 would gain by reporting u'_{m_1}. Thus, only points in the shaded area are possible.

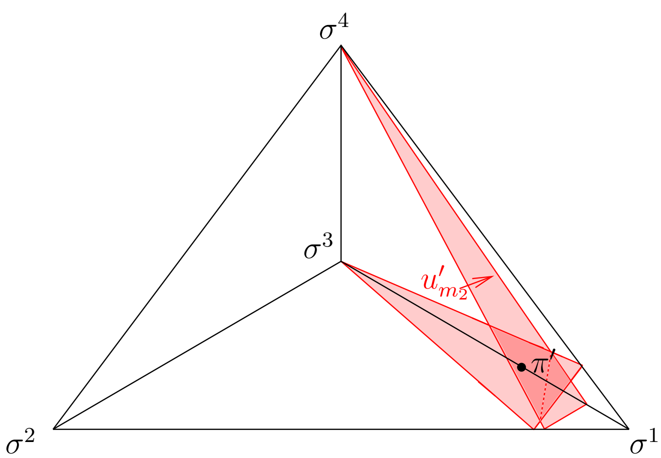
Figure 1: Illustration of the proof of Theorem 1 for the case without money.



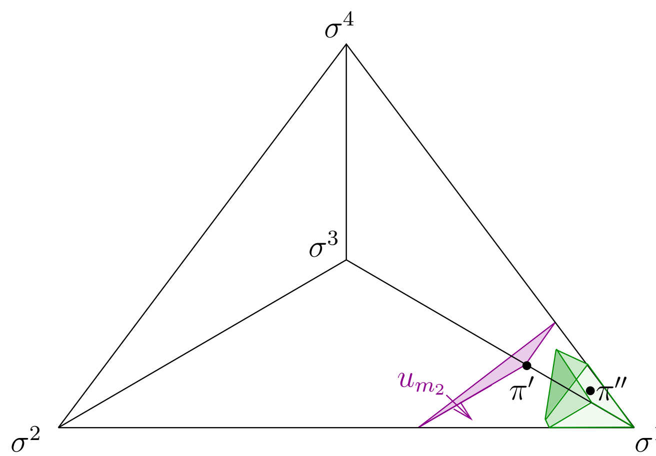
(a) In terms of u_{m_2}, the best choice of φ in the shaded area maximizes the weight on σ^1. We mark it by π'. Since $l \leq \frac{1}{2}$, the weight π' places on σ^1 is no greater than $\frac{u_{m_1w_1}+u_{m_1w_2}}{2u_{m_1w_1}}$. We define u'_{m_1} with indifference lines as depicted. We will construct a profitable misreport for m_2 at (u'_{m_1}, u_{-m_1}).



(b) At (u'_M, u_W), ex-post Pareto efficiency in matching requires φ to choose a convex combination of σ^1, σ^2, σ^3, and σ^4. The indifference plane for u'_{m_1} indicates indifference to shifting mass between σ^2 and σ^4, since both of them match m_1 to w_2 fully. We define u_{m_2} with the indifference planes as depicted.



(c) We define u'_{m_2} with indifference planes as depicted. Ex ante individual rationality requires that $\pi'' \equiv \varphi(u'_M, u_W)$ lie above the indifference plane of u'_{m_2} through σ^4, which leaves m_2 fully unmatched. According to u'_{m_2}, π' is worse than being fully unmatched.



(d) Ex ante individual rationality for m_1 at $u_{m'_1}$ and for m_2 at u'_{m_2} narrows down the possible locations for $\pi'' = \varphi(u'_M, u_W)$ to the bottom-right wedge that includes σ^1, shaded in green. However, according to u_{m_2}, every point in this wedge is better than π'. In turn, as per Figure 2a, π' is no worse than $\varphi(u'_{m_1}, u_{-m_1})$. This means that φ is not strategy-proof, since m_2 gains by reporting u'_{m_2} at the true profile (u'_{m_1}, u_{-m_1}).

Figure 2: Illustration of the proof of Theorem 1 for the case without money, continued.

unmatched than from being matched to someone on the same "side." On this subdomain, the requirement of ex post Pareto efficiency in matching rules out any matching of agents on the same "side," leaving only those matchings that are feasible in the constructed bipartite model. Then the impossibility result of Theorem 1 applies to this subdomain in the non-bipartite model, and so extends to the whole domain. We prove the remaining case where $|N| = 3$ in Appendix A.

If we further drop the assumption that matches are pairwise, then we admit coalition formation problems. Pairwise and (non-)bipartite matching problems can be modeled as coalition formation problems where all agents derive lower utility from being in any coalition with three or more members than from being alone. Ex post Pareto efficiency in matching would rule out any partition of agents where some coalition contains three or more agents. Therefore, Theorems 1 and 2 extend the negative result to coalition formation problems with three or more agents.

Corollary 1. *For fractional coalition formation problems, if $|N| \geq 3$ and $\mathcal{U} \supseteq \mathcal{U}^{slin}$, then no rule for $\mathcal{U}$ is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational.*

Independence of axioms For completeness, we show that the axioms in our theorems are independent by demonstrating the existence of a rule satisfying every pair from the triplet of axioms. If $\Omega > 0$, then for any $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_{++}$ satisfying $\sum_{i \in N} \omega_i = \Omega$, any rule that, for each $u \in \mathcal{U}^{slin}$, selects some *DIP-equilibrium* allocation at money endowment ω (Manjunath, 2016) is fully Pareto efficient (and hence ex post Pareto efficient in matching) and ex ante individually rational. If $\Omega = 0$, then any rule that selects a limit—as $\varepsilon \to 0$—of ε*DIP-equilibrium* allocations (Manjunath, 2016) is an example of such a rule. The "no trade" rule, which leaves every agent unmatched and does not distribute any money to agents, is strategy-proof and ex post (and therefore ex ante) individually rational, but not ex post Pareto efficient in matching. Finally, any serial dictatorship over $\mathcal{Z}$ is strategy-proof and fully Pareto efficient (and hence ex post Pareto efficient in matching) but not ex ante individually rational.

Ordinal rules in the absence of money Without money distribution, an interesting restriction that we may impose on a rule is that it be invariant to all information other than how agents rank their possible partners. A rule φ with is *ordinal* if it only depends on induced preferences over partners. That is, for each pair $u, u' \in \mathcal{U}^{slin}$, if for each $i \in N$, $R(u_i) = R(u'_i)$, then $\varphi(u) = \varphi(u')$.

Ordinality is particularly relevant when the model has as a primitive each agent's preferences over partners. For each $i \in N$, let $\mathcal{P}_i$ be the set of linear orders over J_i. For each $i \in N$ and each $\succeq_i \in \mathcal{P}_i$, let $\mathcal{U}(\succeq_i) \subseteq \mathcal{U}_i^{slin}$ be the set of utility functions that induce $\succeq_i$. If the starting point is a profile of preferences, we may describe a problem by $\succeq \in \times_{i \in N} \mathcal{P}_i \equiv \mathcal{P}$. To consider ordinal rules as defined above is to consider *ranking-based* rules, that is, rules that map $\mathcal{P}$ to Π. A ranking-based rule φ is *deterministic* if for each $\succeq \in \mathcal{P}$, $\varphi(\succeq) \in \Sigma$.

For a bipartite model, Alcalde and Barberà (1994) show that no ranking-based and deterministic rule is strategy-proof, Pareto efficient and *ex post* individually rational.[20] Does dropping the restriction that the rule be deterministic allow an escape from this incompatibility? We define particular extensions of strategy-proofness and individual rationality to non-deterministic rules and explain how Theorem 1 answers this question in the negative.

For ranking-based and non-deterministic rules, a natural generalization of strategy-proofness is *stochastic-dominance (SD) strategy-proofness.* It says that any change that an agent can effect by misreporting their preferences is a worsening in the sense of first-order stochastic dominance. Equivalently, a ranking-based rule φ is SD strategy-proof if for each $\succeq \in \mathcal{P}$, $i \in N$, $\succeq_i' \in \mathcal{P}_i$, and $u_i \in \mathcal{U}(\succeq_i)$,

$$u_i(\varphi_i(\succeq)) \geq u_i(\varphi_i(\succeq_i', \succeq_{-i})).$$

If a ranking-based rule is SD strategy-proof, then it naturally defines a strategy-proof and ordinal rule for $\mathcal{U}^{slin}$.

Similarly, *SD individual rationality* says that each agent receives a lottery that stochastically dominates being alone with certainty. However, this is equivalent to *ex post* individual rationality and therefore implies ex ante individual rationality.

Notice that ranking-based rules are a subset of the rules we have considered in our main result, since they uniquely correspond to ordinal rules for $\mathcal{U}^{slin}$. We thus have the following corollary to Theorem 1, which extends the result of Alcalde and Barberà (1994) to non-deterministic matching.

Corollary 2. *For a bipartite model without money, if $|M| \geq 2$ and $|W| \geq 2$, then no ranking-based rule is SD strategy-proof, ex post Pareto efficient, and SD individually rational.*

Sönmez (1999) shows a general result that for non-bipartite problems implies there is no ranking-based and deterministic rule that is strategy-proof, Pareto efficient and *ex post* individually rational. We extend this result to non-deterministic rules in the following corollary to Theorem 2.

[20] A ranking-based and deterministic rule φ is strategy-proof if for each $i \in N$, $\succeq \in \mathcal{P}$ and $\succeq_i' \in \mathcal{P}_i$, $\varphi_i(\succeq) \succeq_i \varphi_i(\succeq_i', \succeq_{-i})$, and individually rational if $\varphi_i(\succeq) \succeq_i i$.

Corollary 3. *For a non-bipartite model without money, if $|N| \geq 3$, then no ranking-based rule is SD strategy-proof, ex post Pareto efficient, and SD individually rational.*

Lower-bounds on consumption of money We have imposed the restriction that each agent consumes a non-negative quantity of money. Consequently, ours is not a transferable utility model: utility is transferable only to the extent of re-allocating the Ω units of money in the economy, subject to the non-negativity constraint. We interpret these lower bounds as modeling budget constraints on the parts of the agents since our analysis is not sensitive to the lower bound being zero, rather than some other finite amount.[21]

If we drop the non-negativity restriction on the allocation of money, then utility is fully transferable. In this case, full Pareto efficiency implies *allocative efficiency*, which says that the chosen matching maximizes the sum of agents' utilities. However, allocative efficiency is still considerably stronger than ex post Pareto efficiency of the matching rule. Take for example a problem with two agents $i, j \in N$ such that

$$\begin{aligned} u_i(\pi_i, z_i) &= \pi_{ij} + z_i, \text{ and} \\ u_j(\pi_j, z_j) &= -\tfrac{\pi_{ji}}{2} + z_j, \end{aligned}$$

Allocative efficiency says that the two agents are matched fully. On the other hand, every matching in Π is ex post Pareto efficient, since the agents being left unmatched is Pareto efficient. So, for this economy, allocative efficiency narrows the selection down to a single choice while ex post Pareto efficiency in matching says nothing at all.

For the transferable utility version of our model, that is, without a lower-bound on money consumption to respect, there are strategy-proof rules that satisfy normative properties even stronger than allocative efficiency.[22] However, none of these rules respect both ex ante individual rationality and the aggregate budget constraint on the allocation of money. That is, under the feasibility condition that the budget of Ω units of money consumption be respected, strategy-proofness, ex ante individual rationality, and allocative efficiency are incompatible with two or more agents (Green and Laffont, 1977; Holmström, 1979; Myerson and Satterthwaite, 1983).[23] As we discuss in the next section, for the case of two agents,

[21] Given linearity of utility in money, a situation where $b \in \mathbb{R}_+^N$ is the profile of borrowing limits, $\omega \in \mathbb{R}_+^N$ is the profile of money endowments, and $\omega_0 \in \mathbb{R}_+$ is the planner's money endowment can be translated to a situation where money consumption must be non-zero, $\omega + b$ is the profile of money endowments, and $\Omega = \omega_0 + \sum_{i \in N}(b_i + \omega_i)$ is the total amount money in the economy. Recall that our definition of individual rationality makes no reference to any private money endowment of an agent. So if negative money consumption and borrowing up to the limit were allowed this definition requires that each agent i should receive at least as much utility from the rule as they would receive if they were unmatched and with money consumption equal to $-b_i$. See also footnote 18.

[22] See, for instance, Yenmez (2013, 2015).

[23] Though Vickrey-Clarke-Groves rules are strategy-proof and allocatively efficient (Vickrey, 1961; Clarke,

strategy-proofness, ex ante individual rationality and ex post Pareto efficiency in matching *are* compatible. The reason this does not contradict these classical impossibilities for transferable utility models is the weakness of ex post Pareto efficiency in matching as compared to allocative efficiency.

4 Possibility

We discuss how modifying key features of our model permits one to escape impossibility. As we have noted on Page 7, any rule that is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational for problems where $\Omega = 0$ extends trivially to problems where $\Omega > 0$: dispose of the money and then apply the rule. If full Pareto-efficiency is desired as well, it can easily be achieved, by distributing all available money in some fixed manner. Thus, we only consider in this section the case of $\Omega = 0$.

Diversity of preferences The preferences in $\mathcal{U}^{slin}$ include, for each $i \in N$, all orderings over i's partners. The diversity of these preferences for all agents is critical for the incompatibility. In the bipartite model, certain restrictions on the domain of preferences for one of the two sides can alleviate this impossibility. We present here just one such restriction and show the existence of a ranking-based rule.

Suppose that for each agent $m \in M$, their ranking of all agents in W is fixed: for each pair $P_m, P'_m \in \mathcal{P}_m$, and each pair $w, w' \in W$, $w\ P_m\ w'$ if and only if $w\ P'_m\ w'$. Moreover, this fixed ranking is the same across all agents in M: for each pair $m, m' \in M$, each $P_m \in \mathcal{P}_m$, each $P_{m'} \in \mathcal{P}_{m'}$, and each pair $w, w' \in W$, $w\ P_m\ w'$ if and only if $w\ P_{m'}\ w'$. The only variation in preferences of each $m \in M$ is in how they rank being alone. This restriction is meaningful when the ranking of agents in W is objectively fixed, but each agent in M has a different outside option or opportunity cost, which is their private information.[24] Denote by $\mathcal{P}^c_M$ these preferences where the private information of each agents in M is only their cutoff.

Consider the domain $(\mathcal{P}^c_M, \mathcal{P}_W)$. It follows directly from the lattice structure of the core that there is a unique core allocation for each preference profile in this domain. Thus, the rule that selects this unique core allocation defines a strategy-proof, ex post Pareto efficient in matching, and ex post (and hence ex ante) individually rational rule (Sönmez, 1999).

1971; Groves, 1973), they do not satisfy individual rationality on our domain of preferences.

[24] Real-world applications include college admissions when an objective ranking is generated by grade point averages or entrance exam scores, and residency matching when residency positions are commonly ranked by doctors on the basis of prestige or reputation. In addition to cardinal information, the heterogeneity across colleges is in their acceptability cutoff, and across doctors is in their outside option.

Proposition 1. *There exists a rule $\varphi : (\mathcal{P}^c_M, \mathcal{P}_W) \to \Pi$, that is strategy-proof, ex ante Pareto efficient, and ex post individually rational, as well as deterministic and ranking-based.*

While the domain $(\mathcal{P}^c_M, \mathcal{P}_W)$ involves a restriction in preferences across agents, another path to possibility is to restrict each agent to a very sparse set of preferences. Alcalde and Barberà (1994) provide a very severe restriction of this type that is sufficient for the existence of a rule satisfying the properties listed in Proposition 1.

More than one agent on both sides in bipartite model Our impossibility for the bipartite model relies upon multiplicity of agents on both sides. If either $|M| = 1$ or $|W| = 1$, then our axioms turn out to be compatible.

Suppose without loss of generality that $M = \{m\}$. Regardless of cardinality of W, there is a strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational rule φ defined over $\mathcal{U}^{lin}$ as follows: Given $u \in \mathcal{U}^{lin}$, let $\tilde{W}(u) \equiv \{w \in W : u_w(\delta^m_w, 0) \geq u_w(\delta^w_w, 0)$ and $u_m(\delta^w_m, 0) \geq u_m(\delta^m_m, 0)\}$ and, if $\tilde{W}(u) \neq \varnothing$, let $\tilde{w}(u) \in \text{argmax}_{w \in \tilde{W}(u)} u_m(\delta^w_m, 0)$. Finally let, $\varphi(u) \equiv (\sigma, 0)$ where $\sigma \in \Sigma$ is such that if $\tilde{W}(u) \neq \varnothing$, $\sigma_{m\tilde{w}(u)} = 1$ but if $\tilde{W}(u) = \varnothing$, for each $i \in N, \sigma_{ii} = 1$. In words, φ matches m to their most preferred partner among those who prefer being matched to m than being alone. If no such partner exists, it leaves m unmatched. An analogous rule exists if $|W| = 1$.

Proposition 2. *If either $|M| = 1$ or $|W| = 1$, then there exists a rule that is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational.*

Expected utility preferences We have modeled agents as being von Neumann-Morgenstern expected utility maximizers. As noted earlier in the discussion of ordinal rules, first-order stochastic dominance with respect to an ordering over deterministic outcomes defines an incomplete order over lotteries, and every expected utility preference is an extension of some such stochastic dominance order. There are many alternatives to expected utility theory that extend stochastic dominance, but the impossibility result holds for any such alternative that nests expected utility. The lexicographic extension approach (Cho, 2014; Schulman and Vazirani, 2012) is an interesting alternative that does not nest expected utility but extends stochastic dominance.[25]

A *lexicographic extension* preference R^L_i of an agent i over the set of fractional assignments Δ_i is characterized by a complete linear order $\succeq_i$ of i's possible partners, J_i, such that, for

[25] The lexicographic extension is a special case of the more general lexicographic expected utility theory (Hausner, 1954; Thrall, 1954; Chipman, 1960). For allocation problems with lexicographic extension preferences, there exist strategy-proof, Pareto efficient, and envy-free rules for the case of divisible goods (Schulman and Vazirani, 2012) and of indivisible goods (Cho, 2018; Saban and Sethuraman, 2014). For more on lexicographic extensions, we refer the reader to Cho (2012), Cho (2014), and Cho and Doğan (2016).

each pair $\pi_i, \pi'_i \in \Delta_i$, $\pi_i \; R_i^L \; \pi'_i$ if and only if $\pi_i = \pi'_i$ or there is $k \in J_i$ such that for each $l \succ_i k$, $\pi_{il} = \pi'_{il}$ and $\pi_{ik} > \pi'_{ik}$.

Let $\mathcal{R}_i^L$ be the set of all lexicographic extension preferences on Δ_i. Let $\mathcal{R}^L \equiv \times_{i \in N} \mathcal{R}_i^L$. A rule φ maps $\mathcal{R}^L$ to Π. Given that each $R_i^L \in \mathcal{R}_i^L$ uniquely corresponds to a linear order $\succeq_i$ over possible partners of i, its *partner ranking*, a deterministic matching is Pareto efficient with respect to R^L if and only if it is Pareto efficient with respect to $\succeq$. So, $\pi \in \Pi$ is ex post Pareto efficient at R^L if and only if the deterministic matchings in the support of π are Pareto efficient with respect to the corresponding $\succeq$. The appropriate definitions of strategy-proofness and ex ante individual rationality use R_i^L to compare lotteries. A rule φ on $\mathcal{R}^L$ is *strategy-proof* if for each $R^L \in \mathcal{R}^L$, each $i \in N$, and each $\tilde{R}_i^L \in \mathcal{R}_i^L$, $\varphi_i(R^L) \; R_i^L \; \varphi_i(\tilde{R}_i^L, R_{-i}^L)$. It is ex ante individually rational if for each $R^L \in \mathcal{R}^L$ and each $i \in N$, $\varphi_i(R^L) \; R_i^L \; \delta_i^i$. Notice that $\pi \in \Pi$ is ex ante individually rational at R^L if and only if for each $i \in N$, $\pi_i = \delta_i^i$ or $\pi_{ij} > 0$ for some partner $j \in J_i \setminus \{i\}$ such that $j \succeq_i i$, where $\succeq_i$ is the linear order corresponding to R_i^L.

We describe a class of rules Φ^L for lexicographic extension preferences that satisfy our target properties of strategy-proofness, ex post Pareto efficiency in matching, and ex ante individual rationality. These rules are defined by a procedure resembling randomization over serial dictatorships.

Let $\mathcal{O}$ be the set of all ($|N|!$) orderings of the agents in N. Each member of Φ^L is parameterized by an exogenous randomization device ρ that has the following *full support property*: for each $i \in N$, there is a $o \in \mathcal{O}$ that puts i first and has $\rho(o) > 0$. Fix such a randomization device ρ. The following procedure uses ρ to define a rule $\varphi \in \Phi^L$. For each $o \in \mathcal{O}$ and each preference profile R^L, it produces a deterministic matching, and $\varphi(R^L)$ puts probability $\rho(o)$ on it.

1. **Randomize:** Use randomization device ρ to determine $o \in \mathcal{O}$.
2. **Initialize:** Begin with no agent *marked* as "removed".
3. **Removal Phase:** *Select* the highest ranked *unmarked* agent, according to o, amongst those who find no other *unmarked* agent acceptable. Match this agent to themselves, and *mark* them. Repeat until there is no such agent.
4. **Match Phase:** *Select* the highest ranked *unmarked* agent, according to o. Match this agent to their most preferred partner amongst those *unmarked*, and *mark* them both. Repeat until no unmarked agent remains.

We prove the following result in Appendix A.

Proposition 3. *Each rule in Φ^L is strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational.*

Appendices

A Proofs

Proof of Theorem 1. We prove the result by contradiction. Suppose that φ is a strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational rule. Let $K > 12\Omega$.

We start with the case where $|M| = |W| = 2$. Let $M \equiv \{m_1, m_2\}$ and $W \equiv \{w_1, w_2\}$. Let $u \in \mathcal{U}^{slin}$ be such that for each $(\pi, z) \in \mathcal{Z}$,

$$\begin{aligned}
u_{m_1}(\pi_{m_1}, z_{m_1}) &= K\pi_{m_1w_1} + \tfrac{K}{2}\pi_{m_1w_2} + z_{m_1},\\
u_{m_2}(\pi_{m_2}, z_{m_2}) &= K\pi_{m_2w_2} + \tfrac{K}{2}\pi_{m_2w_1} + z_{m_2},\\
u_{w_1}(\pi_{w_1}, z_{w_1}) &= K\pi_{w_1m_2} + \tfrac{K}{2}\pi_{w_1m_1} + z_{w_1}, \text{ and}\\
u_{w_2}(\pi_{w_2}, z_{w_2}) &= K\pi_{w_2m_1} + \tfrac{K}{2}\pi_{w_2m_2} + z_{w_2}.
\end{aligned}$$

Let $(\pi, z) \equiv \varphi(u)$. By definition of u, the only Pareto efficient deterministic matchings in Σ are σ^1 and σ^2, where $\sigma^1_{m_1w_1} = \sigma^1_{m_2w_2} = 1$ and $\sigma^2_{m_1w_2} = \sigma^2_{m_2w_1} = 1$. By ex post Pareto efficiency, π is a convex combination of σ^1 and σ^2. Let $l \in [0,1]$ be such that $\pi = l\sigma^1 + (1-l)\sigma^2$.

Suppose that $l \le \frac{1}{2}$. Let $\alpha > \frac{14K+32\Omega}{K-12\Omega}$.[26] By definition of Ω and K, $\frac{14K+32\Omega}{K-12\Omega} \ge 14$, and so $\alpha > 14$.[27] Let $u'_{m_1} \in \mathcal{U}^{slin}_{m_1}$ be such that for each $(\pi_{m_1}, z_{m_1}) \in X_{m_1}$,

$$u'_{m_1}(\pi_{m_1}, z_{m_1}) = K\pi_{m_1w_1} - \alpha K\pi_{m_1w_2} + z_{m_1}.$$

Let $(\pi', z') \equiv \varphi(u'_{m_1}, u_{-m_1})$. By definition of (u_{m_1}, u_{-m_1}), the only Pareto efficient deterministic matchings are σ^1 and σ^2 defined above and $\sigma^3 \in \Sigma$ such that $\sigma^3_{m_1m_1} = \sigma^3_{w_2w_2} = \sigma^3_{m_2w_1} = 1$. By ex post Pareto efficiency, π' is a convex combination of σ^1, σ^2, and σ^3. Thus,

$$\pi'_{m_1w_1} = \pi'_{m_2w_2}. \tag{1}$$

Notice that $u_{m_1}(\pi_{m_1}, z_{m_1}) = Kl + K(\frac{1-l}{2}) + z_{m_1} = K(\frac{1+l}{2}) + z_{m_1}$. By strategy-proofness,

$$K\pi'_{m_1w_1} + \frac{K}{2}\pi'_{m_1w_2} + z'_{m_1} = u_{m_1}(\pi'_{m_1}, z'_{m_1}) \le u_{m_1}(\pi_{m_1}, z_{m_1}) = K\left(\frac{1+l}{2}\right) + z_{m_1}.$$

[26] We define α this way to ensure that $\frac{\alpha K-\Omega}{\alpha+2} > \frac{7}{8}K + \frac{3}{2}\Omega$, a fact that we appeal to below.

[27] Since When $\Omega = 0$, K is any positive number and $\alpha > 14$.

Then, since $K > 0$, $z_{m_1} \leq \Omega$, and $z'_{m_1} \geq 0$,

$$K\pi'_{m_1w_1} \leq K\pi'_{m_1w_1} + \frac{K}{2}\pi'_{m_1w_2} \leq K\left(\frac{1+l}{2}\right) + z_{m_1} - z'_{m_1} \leq K\left(\frac{1+l}{2}\right) + \Omega.$$

Thus, since $l \leq \frac{1}{2}$ and $K > 12\Omega$,

$$\pi'_{m_1w_1} \leq \frac{3}{4} + \frac{\Omega}{K} < 1. \tag{2}$$

Since $\pi'_{m_2m_2} = 0$ and $z'_{m_2} \leq \Omega$, by (1) and (2),

$$u_{m_2}(\pi'_{m_2}, z'_{m_2}) \leq K\left(\frac{3}{4} + \frac{\Omega}{K}\right) + \frac{K}{2}\left(\frac{1}{4} - \frac{\Omega}{K}\right) + \Omega = \frac{7}{8}K + \frac{3}{2}\Omega. \tag{3}$$

Let $u'_{m_2} \in \mathcal{U}^{slin}_{m_2}$ be such that for each $(\pi_{m_2}, z_{m_2}) \in X_{m_2}$,

$$u'_{m_2}(\pi_{m_2}, z_{m_2}) = K\pi_{m_2w_2} - \alpha K\pi_{m_2w_1} + z_{m_2}.$$

Define $(\pi'', z'') \equiv \varphi(u'_M, u_W)$. By definition of (u'_M, u_W), the only Pareto efficient deterministic matchings are σ^1, σ^2, and σ^3 defined above and $\sigma^4 \in \Sigma$ such that $\sigma^4_{m_1w_2} = \sigma^4_{m_2m_2} = \sigma^4_{w_1w_1} = 1$. By ex post Pareto efficiency, π'' is a convex combination of $\sigma^1, \sigma^2, \sigma^3$, and σ^4. So there are $p, q, r, s \in [0,1]$ such that $\pi'' = p\sigma^1 + q\sigma^2 + r\sigma^3 + s\sigma^4$ and $p + q + r + s = 1$.

Since (π'', z'') is ex ante individually rational at (u'_M, u_W) and $p = 1 - q + r + s$,

$$u'_{m_1}(\pi''_{m_2}, z''_{m_1}) = Kp - \alpha K(q+s) + z''_{m_1} \geq 0 = u'_{m_1}(\delta^{m_1}_{m_1}, 0) \tag{4}$$

and

$$u'_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp - \alpha K(q+r) + z''_{m_2} \geq 0 = u''_{m_2}(\delta^{m_2}_{m_2}, 0) \tag{5}$$

Adding (4) and (5),

$$2Kp - \alpha K(q+r+s) - \alpha Kq + z''_{m_1} + z''_{m_2} \geq 0,$$

which, since $p = 1 - q - r - s$, implies

$$2Kp - \alpha K(1-p) \geq \alpha Kq - z''_{m_1} - z''_{m_2}.$$

Then, since $\alpha K q \geq 0$ and $-z''_{m_1} - z''_{m_2} \geq -\Omega$, we have $2Kp - \alpha K(1-p) \geq -\Omega$, so

$$p \geq \frac{\alpha K - \Omega}{(\alpha+2)K}. \tag{6}$$

By definition of K and α, $\alpha K > \Omega$. Thus, the right hand side of (6) is positive but less than one.

Finally, since $K > 0$ and $z''_{m_2} \geq 0$, and by (6),

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp + \frac{K}{2}(q+r) + z''_{m_2} \geq Kp \geq \frac{\alpha K - \Omega}{\alpha + 2}.$$

However, by definition of α and (3),

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) \geq \frac{\alpha K - \Omega}{\alpha + 2} > \frac{7}{8}K + \frac{3}{2}\Omega \geq u_{m_2}(\pi'_{m_2}, z'_{m_2}).$$

This contradicts the strategy-proofness of φ, since m_2 can profitably manipulate it by reporting u'_{m_2} at the profile (u'_{m_1}, u_{-m_1}).

Thus, $l > \frac{1}{2}$. However, we reach an analogous contradiction by interchanging the roles of M and W.

Now, we consider the case of $|M| > 2$ or $|W| > 2$. Since both $|M| \geq 2$ or $|W| \geq 2$, let m_1 and m_2 be distinct members of M and let w_1 and w_2 be distinct members of W. Let $m_3, m_4, \ldots$ be a labeling of $M \setminus \{m_1, m_2\}$ and $w_3, w_4, \ldots$ be a labeling of $W \setminus \{w_1, w_2\}$. We start with $u \in \mathcal{U}^{slin}$ such that for each $(\pi, z) \in Z$,

$$\begin{array}{rcl}
u_{m_1}(\pi_{m_1}, z_{m_1}) & = & K\pi_{m_1w_1} + \frac{K}{2}\pi_{m_1w_2} - \sum_{t=3}^{|W|} t\pi_{m_1w_t} + z_{m_1}, \\
u_{m_2}(\pi_{m_2}, z_{m_2}) & = & K\pi_{m_2w_2} + \frac{K}{2}\pi_{m_2w_1} - \sum_{t=3}^{|W|} t\pi_{m_2w_t} + z_{m_2}, \\
u_{w_1}(\pi_{w_1}, z_{w_1}) & = & K\pi_{w_1m_2} + \frac{K}{2}\pi_{w_1m_1} - \sum_{t=3}^{|M|} t\pi_{w_1m_t} + z_{w_1}, \\
u_{w_2}(\pi_{w_2}, z_{w_2}) & = & K\pi_{w_2m_1} + \frac{K}{2}\pi_{w_2m_2} - \sum_{t=3}^{|M|} t\pi_{w_2m_t} + z_{w_2},
\end{array}$$

for each $m \in M \setminus \{m_1, m_2\}$,

$$u_m(\pi_m, z_m) = z_m - \sum_{t=1}^{|W|} t\pi_{mw_t},$$

and for each $w \in W \setminus \{w_1, w_2\}$,

$$u_w(\pi_w, z_w) = z_w - \sum_{t=1}^{|M|} t\pi_{wm_t}.$$

By definition of u, for each Pareto efficient deterministic matching σ and each distinct pair $i, j \in N$ such that $i \notin \{m_1, m_2, w_1, w_2\}$, $\sigma_{ij} = 0$. Thus, for each ex post Pareto efficient $\pi \in \Pi$ and each $i \in N \setminus \{m_1, m_2, w_1, w_2\}$, $\pi_{ii} = 1$. Since each agent in $N \setminus \{m_1, m_2, w_1, w_2\}$ remains unmatched at any ex post Pareto efficient matching, the proof proceeds in the same manner as the case with $|M| = |W| = 2$. $\square$

Proof of Theorem 2, Case $|N| = 3$. Since we restrict attention to the case of three agents, we name them 1,2, and 3.

For this three agent model, Σ consists of the following four deterministic matchings:

$$\sigma^0 = \begin{pmatrix} 1 \leftrightarrow 1 \\ 2 \leftrightarrow 2 \\ 3 \leftrightarrow 3 \end{pmatrix}, \sigma^{12} = \begin{pmatrix} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 1 \\ 3 \leftrightarrow 3 \end{pmatrix}, \sigma^{13} = \begin{pmatrix} 1 \leftrightarrow 3 \\ 2 \leftrightarrow 2 \\ 3 \leftrightarrow 1 \end{pmatrix}, \sigma^{23} = \begin{pmatrix} 1 \leftrightarrow 1 \\ 2 \leftrightarrow 3 \\ 3 \leftrightarrow 2 \end{pmatrix}.$$

The set of fractional matchings Π is the convex hull of these four deterministic matchings.

We now present a series of claims before completing the proof. Our first claim is a consequence of the following: if $\pi \in \Pi$ is ex post Pareto efficient, then it places zero weight on elements of Σ that are Pareto dominated.

Claim 1. *If π is ex post Pareto efficient at $u \in \mathcal{U}^{slin}$ and there is a distinct pair of agents $i, j \in N$ such that i most prefers j under $R(u_i)$ and j most prefers i under $R(u_j)$, then π puts zero weight on σ^0.*

Our remaining claims bound from below the weight that is placed on matching a distinct pair of agents who prefer each other over the third. Whether such a bound exists, and the magnitude of this bound when it does, depends on the marginal rate of substitution between the matching portion of each agent's consumption and money. In what follows, suppose there exists a strategy-proof, ex post Pareto efficient in matching, and ex ante individually rational rule φ.

Claim 2. *Let $K > 3\Omega$. Suppose $u \in \mathcal{U}^{slin}$ is such that there is a distinct pair $i, j \in N$ for whom*

$$u_i(\pi_i, z_i) = K\pi_{ij} - K\pi_{ik} + z_i$$
$$\textit{and}$$
$$u_j(\pi_j, z_j) = K\pi_{ji} - K\pi_{jk} + z_j,$$

where $\{k\} = N \setminus \{i, j\}$. If $(\pi, z) = \varphi(u)$, then $\pi_{ij} \geq 1 - \frac{3\Omega}{K}$.

Proof of Claim 2. We fix $u_k \in \mathcal{U}_k^{slin}$ and prove that as long as u_i and u_j satisfy the conditions in the claim, then they are matched to one another within $\frac{3\Omega}{K}$. Consequently, we denote a profile of utilities only as a pair of utility functions, one for each of i and j.

Let $u^0 \in \mathcal{U}^{slin}$ be such that

$$u_i^0(\pi_i, z_i) = K\pi_{ij} - K\pi_{ij} + z_i$$
and
$$u_j^0(\pi_j, z_j) = K\pi_{ji} - K\pi_{jk} + z_j.$$

Let $(\pi^{00}, z^{00}) \equiv \varphi(u_i^0, u_j^0)$. Suppose that $\pi_{ij}^{00} < 1 - \frac{3\Omega}{K}$. Then by ex post Pareto efficiency and Claim 1, $\pi_{ik}^{00} > 0$ or $\pi_{jk}^{00} > 0$. Without loss of generality, suppose that $\pi_{ik}^{00} > 0$.

Let $\alpha > 0$ be such that

$$1 > \frac{\alpha - \frac{\Omega}{K}}{\alpha + 2} > \frac{\frac{\pi_{ij}^{00}+1}{2} + \frac{\Omega}{2K} + 1}{2} + \frac{\Omega}{2K}.$$

Such α exists since $\pi_{ij}^{00} < 1 - \frac{3\Omega}{K}$. Let $u^1 \in \mathcal{U}^{slin}$ be such that

$$u_i^1(\pi_i, z_i) = K\pi_{ij} - \alpha K\pi_{ik} + z_i$$
and
$$u_j^1(\pi_j, z_j) = K\pi_{ji} - \alpha K\pi_{jk} + z_j.$$

Let $(\pi^{10}, z^{10}) \equiv \varphi(u_i^1, u_j^0), (\pi^{01}, z^{01}) \equiv \varphi(u_i^0, u_j^1)$, and $(\pi^{11}, z^{11}) \equiv \varphi(u_i^1, u_j^1)$. Since φ is strategy-proof, i does not benefit by misreporting u_i^1 at the preference profile u^0, so

$$\begin{array}{rrcll} & K\pi_{ij}^{10} - K\pi_{ik}^{10} + z_i^{10} & \leq & K\pi_{ij}^{00} - K\pi_{ik}^{00} + z_i^{00} & [\text{since } u_i^0(\pi_i^{10}, z_i^{10}) \leq u_i^0(\pi_i^{00}, z_i^{00})] \\ \Rightarrow & K\pi_{ij}^{10} - K(1 - \pi_{ij}^{10}) + 0 & \leq & K\pi_{ij}^{00} - 0 + \Omega & [\text{since } \pi_{ik}^{10} \leq 1 - \pi_{ij}^{10}, \pi_{ik}^{00} \geq 0, \\ & & & & z_i^{10} \geq 0, \text{ and}, z_i^{00} \leq \Omega] \end{array}$$

Thus,

$$\pi_{ij}^{10} \leq \frac{\pi_{ij}^{00} + 1}{2} + \frac{\Omega}{2K} \tag{7}$$

Since φ is ex ante individually rational, $u_i^1(\pi_i^{11}, z_i^{11}) \geq 0$ and $u_j^1(\pi_j^{11}, z_j^{11}) \geq 0$. That is,

$$K\pi_{ij}^{11} - \alpha K\pi_{ik}^{11} + z_i^{11} \geq 0$$
and
$$K\pi_{ji}^{11} - \alpha K\pi_{jk}^{11} + z_j^{11} \geq 0.$$

Adding these together, and by symmetry of π^{11} (that is, $\pi_{ij}^{11} = \pi_{ji}^{11}$), by Claim 1 (that is, $\pi_{ij}^{11} + \pi_{jk}^{11} + \pi_{ki}^{11} = 1$), and by budget feasibility of z^{11} (that is, $z_i^{11} + z_j^{11} \leq \Omega$),

$$\begin{array}{rrcl} & 2K\pi_{ij}^{11} - \alpha K(1 - \pi_{ij}^{11}) + \Omega & \geq & 0 \\ \Rightarrow & K(\alpha + 2)\pi_{ij}^{11} - \alpha K + \Omega & \geq & 0 \end{array}$$

yielding

$$\pi_{ij}^{11} \geq \frac{\alpha - \frac{\Omega}{K}}{\alpha + 2}. \tag{8}$$

From (7), (8), and the definition of α,

$$\begin{array}{rrcll} & \pi_{ij}^{11} & > & \frac{\pi_{ij}^{10}+1}{2} + \frac{\Omega}{2K} & \\ \Rightarrow & 2K\pi_{ij}^{11} - K & > & K\pi_{ij}^{10} + \Omega & \\ \Rightarrow & K\pi_{ji}^{11} - K(1-\pi_{ji}^{11}) & > & K\pi_{ji}^{10} + \Omega & \\ \Rightarrow & K\pi_{ji}^{11} - K\pi_{jk}^{11} & > & K\pi_{ji}^{10} - K\pi_{jk}^{10} + \Omega & [\text{by symmetry of } \pi^{10} \text{ and } \pi^{11}, \text{ Claim 1,} \\ & & & & \text{and } \pi_{jk}^{10} \geq 0] \\ \Rightarrow & K\pi_{ji}^{11} - K\pi_{jk}^{11} + z_j^{11} & > & K\pi_{ji}^{10} - K\pi_{jk}^{10} + z_j^{10} & [\text{since } z_j^{11} \geq 0 \text{ and } z_j^{10} \leq \Omega] \\ \Rightarrow & u_j^0(\pi_j^{11}, z_j^{11}) & > & u_j^0(\pi_j^{10}, z_j^{10}). & \end{array}$$

However, this contradicts the strategy-proofness of φ since j benefits from misreporting u_j^1 at the preference profile (u_i^1, u_j^0). Thus, $\pi_{ij}^{00} \geq 1 - \frac{3\Omega}{K}$. □

Claim 3. *Let $K > 16\Omega$. Suppose $u \in \mathcal{U}^{slin}$ is such that there is a distinct pair $i, j \in N$ for whom*

$$u_i(\pi_i, z_i) = K\pi_{ij} + \tfrac{3K}{4}\pi_{ik} + z_i$$
and
$$u_j(\pi_j, z_j) = K\pi_{ji} - K\pi_{jk} + z_j,$$

where $\{k\} = N \setminus \{i, j\}$. If $(\pi, z) = \varphi(u)$, then $\pi_{ij} \geq 1 - \frac{16\Omega}{K}$.

Proof of Claim 3. Consider $u_i' \in \mathcal{U}_i^{slin}$ such that

$$u_i'(\pi_i, z_i) = K\pi_{ij} - K\pi_{ik} + z_i$$

and let $(\pi', z') = \varphi(u_i', u_{-i})$. By Claim 2, $\pi_{ij}' \geq 1 - \frac{3\Omega}{K}$. Thus, $u_1(\pi_i', z_i') \geq K(1 - \frac{3\Omega}{K}) = K - 3\Omega$.

If $\pi_{ij} < 1 - \frac{16\Omega}{K}$, then since $z_i \in [0, \Omega]$,

$$u_i(\pi_i, z_i) < K\left(1 - \frac{16\Omega}{K}\right) + \frac{3K}{4}\left(\frac{16\Omega}{K}\right) + \Omega = K - 3\Omega.$$

Thus, $u_i(\pi_i, z_i) < u_i(\pi_i', z_i')$, contradicting the strategy-proofness of φ. So, $\pi_{ij} \geq 1 - \frac{16\Omega}{K}$. □

Claim 4. *Let $K > 68\Omega$. Suppose $u \in \mathcal{U}^{slin}$ is such that there is a distinct pair $i, j \in N$ for whom*

$$u_i(\pi_i, z_i) = K\pi_{ij} + \tfrac{3K}{4}\pi_{ik} + z_i$$
and
$$u_j(\pi_j, z_j) = K\pi_{ji} + \tfrac{3K}{4}\pi_{jk} + z_j,$$

where $\{k\} = N \setminus \{i,j\}$. *If* $(\pi, z) = \varphi(u)$, *then* $\pi_{ij} \geq 1 - \frac{68\Omega}{K}$.

Proof of Claim 4. Consider $u'_j \in \mathcal{U}_j^{slin}$ such that

$$u'_j(\pi_j, z_j) = K\pi_{ji} - K\pi_{jk} + z_j$$

and let $(\pi', z') = \varphi(u'_j, u_{-j})$. By Claim 3, $\pi'_{ij} \geq 1 - \frac{16\Omega}{K}$. Thus, $u_j(\pi'_j, z'_j) \geq K(1 - \frac{16\Omega}{K}) = K - 16\Omega$.

If $\pi_{ij} < 1 - \frac{68\Omega}{K}$, then and $z_j \in [0, \Omega]$,

$$u_j(\pi_j, z_j) < K\left(1 - \frac{68\Omega}{K}\right) + \frac{3K}{4}\left(\frac{68\Omega}{K}\right) + \Omega = K - 16\Omega.$$

Thus, $u_j(\pi_j, z_j) < u_j(\pi'_j, z'_j)$, contradicting the strategy-proofness of φ. So, $\pi_{ij} \geq 1 - \frac{68\Omega}{K}$. $\square$

Let $K > 306\Omega$. We now complete the proof of Theorem 2 by considering $u, u' \in \mathcal{U}^{slin}$ such that

$$\begin{aligned}
u_1(\pi_1, z_1) &= K\pi_{12} + \tfrac{3K}{4}\pi_{13} + z_1, \\
u'_1(\pi_1, z_1) &= \tfrac{3K}{4}\pi_{12} + K\pi_{13} + z_1, \\
u_2(\pi_2, z_2) &= K\pi_{23} + \tfrac{3K}{4}\pi_{21} + z_2, \\
u'_2(\pi_2, z_2) &= \tfrac{3K}{4}\pi_{23} + K\pi_{21} + z_2, \\
u_3(\pi_3, z_3) &= K\pi_{31} + \tfrac{3K}{4}\pi_{32} + z_3, \\
\text{and} \quad u'_3(\pi_3, z_3) &= \tfrac{3K}{4}\pi_{31} + K\pi_{32} + z_3,
\end{aligned}$$

Let $(\pi, z) \equiv \varphi(u)$, $(\pi^1, z^1) \equiv \varphi(u'_1, u_{-1})$, $(\pi^2, z^2) \equiv \varphi(u'_2, u_{-2})$, and $(\pi^3, z^3) \equiv \varphi(u'_3, u_{-3})$. By Claim 4, π^1 puts at least $1 - \frac{68\Omega}{K}$ on σ^{13}, π^2 puts at least $1 - \frac{68\Omega}{K}$ on σ^{12}, and π^3 puts at least $1 - \frac{68\Omega}{K}$ on σ^{23}.

Since φ is strategy-proof,

$$\begin{aligned}
K\pi_{12} + \frac{3K}{4}\pi_{13} + z_1 = u_1(\pi_1, z_1) \geq u_1(\pi_1^1, z_1^1) \geq \frac{3K}{4}\left(1 - \frac{68\Omega}{K}\right) = \frac{3K}{4} - 51\Omega \\
K\pi_{23} + \frac{3K}{4}\pi_{21} + z_2 = u_2(\pi_2, z_2) \geq u_2(\pi_2^2, z_2^2) \geq \frac{3K}{4}\left(1 - \frac{68\Omega}{K}\right) = \frac{3K}{4} - 51\Omega \\
K\pi_{31} + \frac{3K}{4}\pi_{32} + z_3 = u_3(\pi_3, z_3) \geq u_3(\pi_3^3, z_3^3) \geq \frac{3K}{4}\left(1 - \frac{68\Omega}{K}\right) = \frac{3K}{4} - 51\Omega
\end{aligned}$$

Since π is symmetric, summing these inequalities,

$$\frac{7K}{4}(\pi_{12} + \pi_{13} + \pi_{23}) \geq \frac{9K}{4} - 153\Omega.$$

However, feasibility requires that $\pi_{12} + \pi_{13} + \pi_{23} \leq 1$. Therefore,

$$K \leq 306\Omega,$$

which is a contradiction. $\square$

Proof of Proposition 3. Let ρ be the randomization device that defines φ.

For each $R^L \in \mathcal{R}^L$, and each $o \in \mathcal{O}$, the procedure begins with all agents and successively matches one agent at a time to their most preferred partner (possibly themselves) who is yet to be matched. Therefore, the deterministic matching it produces, given any $o \in \mathcal{O}$, is Pareto efficient at R^L. Thus, φ, defined at each R^L as a ρ-generated lottery over Pareto-efficient matchings, is ex post Pareto efficient in matching.

Fix $R^L \in \mathcal{R}^L, i \in N$, and $\tilde{R}_i^L \in \mathcal{R}_i^L$. Let $\succeq_i$ and $\tilde{\succeq}_i$ be the partner rankings of R_i^L and $\tilde{R}_i^L$, respectively. Define $\pi = \varphi(R_i^L, R_{-i}^L)$ and $\tilde{\pi} = \varphi(\tilde{R}_i^L, R_{-i}^L)$. Since we have fixed R^L, i, and $\tilde{R}_i^L$ arbitrarily, to show φ is ex ante individually rational and strategy-proof, it suffices to show that $\pi_i \; R_i^L \; \delta_i^i$ and $\pi_i \; R_i^L \; \tilde{\pi}_i$, respectively.

For each $o \in \mathcal{O}$, let $A(o)$ and $\tilde{A}(o)$ be the sets of all agents who are unmarked when the procedure first enters the Match Phase under (R_i^L, R_{-i}^L) and $(\tilde{R}_i^L, R_{-i}^L)$, respectively. For each $o \in \mathcal{O}$, let $B(o)$ and $\tilde{B}(o)$ be the set of all agents who are unmarked just before the procedure matches i under (R_i^L, R_{-i}^L) and $(\tilde{R}_i^L, R_{-i}^L)$, respectively. Let β be the highest-ranked partner of i in $A \cap J_i$ according to $\succeq_i$. Define $\tilde{\beta}$ analogously using $\tilde{A}$ and $\tilde{\succeq}_i$. For each $N' \subseteq N$ and each $o \in \mathcal{O}$, let $\tau(N', o)$ be the first agent in N' according to o.

The following observations follow from the definition of the procedure.

Observation 1. *For each pair $o, o' \in \mathcal{O}$, $A(o) = A(o')$ and $\tilde{A}(o) = \tilde{A}(o')$. Henceforth, we refer to A and $\tilde{A}$ without reference to the order.*

Observation 2. *If $i \notin A$, then $\pi_i = \delta_i^i$. If $i \notin \tilde{A}$, then $\tilde{\pi}_i = \delta_i^i$.*

Observation 3. *If $i \in A \cap \tilde{A}$, then $A = \tilde{A}$.*

Observation 4. *If $i \in \tilde{A} \setminus A$, then for each $j \in \tilde{A}$, $i \succeq_i j$.*

Observation 5. *If $A = \tilde{A}$, then, for each $o \in \mathcal{O}$, $B(o) = \tilde{B}(o)$.*

Claim 1. *If $i \in A$, then $\beta \succ_i i$ and $\pi_{i\beta} > 0$. If $i \in \tilde{A}$, then $\tilde{\pi}_{i\tilde{\beta}} > 0$.*

Proof. By definition of A, $i \in A$ implies there is some $j \in A$ such that $j \succ_i i$. Thus, $\beta \succ_i i$. At each $o \in \mathcal{O}$ that puts i first, i is selected and matched to β in the first step of the Match Phase. By the full support property of ρ, there is at least one such $o \in \mathcal{O}$ selected with probability $\rho(o) > 0$, so $\pi_{i\beta} > 0$. By identical reasoning, if $i \in \tilde{A}$, then $\tilde{\pi}_{i\tilde{\beta}} > 0$. $\square$

We now show that φ is ex ante individually rational. If $i \notin A$, then $\pi_i = \delta_i^i$, by Observation 2. So suppose instead that $i \in A$. By Claim 1, $\beta \succ_i i$ and $\pi_{i\beta} > 0$, so we have $\pi_i \; R_i^L \; \delta_i^i$. Thus, φ is ex ante individual rational.

We now show that φ is strategy-proof. We examine each of the four possible cases.

Case $\boldsymbol{i \notin A \cup \tilde{A}}$: By Observation 2, $\pi_i = \tilde{\pi}_i = \delta_i^i$.

Case $\boldsymbol{i \in A \setminus \tilde{A}}$: Since $i \in A$, by Claim 1, $\beta \succ_i i$ and $\pi_{i\beta} > 0$. Since $i \notin \tilde{A}$, by Observation 2, $\tilde{\pi}_i = \delta_i^i$. Thus, $\pi_i \; P_i^L \; \tilde{\pi}_i = \delta_i^i$.

Case $\boldsymbol{i \in \tilde{A} \setminus A}$: Since $i \notin A$, by Observation 2, $\pi_i = \delta_i^i$. Since $i \in \tilde{A}$, by Claim 1, $\tilde{\pi}_{i\tilde{\beta}} > 0$. By Observation 4, for each $j \in \tilde{A}$, in particular for $j = \tilde{\beta} \neq i$, $i \succeq_i j$. Thus, for each $j \in N$ such that $\tilde{\pi}_{ij} > 0$, $i \succeq_i j$ and for $j = \tilde{\beta}$, $\tilde{\pi}_{ij} > 0$ and $i \succ_i j$. Thus, $\delta_i^i = \pi_i \; P_i^L \; \tilde{\pi}_i$.

Case $\boldsymbol{i \in A \cap \tilde{A}}$: Let $o \in \mathcal{O}$. Recall, $B(o)$ and $\tilde{B}(o)$ are the sets of unmarked agents just before the step in which i is matched, when i reports $\succeq_i$ and $\tilde{\succeq}_i$, respectively. At $\succeq_i$, let k be the agent *selected* at this step using o. By Observations 3 and 5, $A = \tilde{A}$ and $B(o) = \tilde{B}(o)$. So k is also *selected* at this step at $\tilde{\succeq}_i$. Let $j_o \in B(o)$ and $\tilde{j}_o \in B(o)$ be the agents to whom the procedure matches i, at $\succeq_i$ and $\tilde{\succeq}_i$, respectively. If $k \neq i$, then $k = j_o = \tilde{j}_o$, since k's reported preference is fixed. Instead, if $k = i$, j_o is the highest-ranked partner of i in $B(o) \cap J_i$ according to $\succeq_i$, so $j_o \succeq_i \tilde{j}_o$. Since for each $o \in \mathcal{O}$, $j_o \succeq_i \tilde{j}_o$, we conclude $\pi_i \; R_i^L \; \tilde{\pi}_i$. $\square$

References

Alcalde, José and Salvador Barberà (1994) "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory*, Vol. 4, No. 3, pp. 417–35, May. [2], [11], [14]

Birkhoff, Garrett (1946) "Three observations on linear algebra," *Universidad Nacional de Tucuman, Revista Ser. A*, Vol. 5, pp. 146–151. [5]

Bogomolnaia, Anna and Hervé Moulin (2001) "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, Vol. 100, pp. 295–328. [3]

——— (2004) "Random Matching under Dichotomous Preferences," *Econometrica*, Vol. 72, No. 1, pp. 257–279, Jan. [2]

Chang, Hee-In and Youngsub Chun (2016) "Probabilistic assignment of indivisible objects when agents have single-peaked preferences with a common peak," working paper, Seoul National University. [3]

Chipman, John S. (1960) "The Foundations of Utility," *Econometrica*, Vol. 28, No. 2, pp. 193–224, April. [14]

Cho, Wonki Jo (2012) "On the extension of preferences over objects to preferences over lotteries," working paper, Korea University. [14]

——— (2014) "Impossibility results for parametrized notions of efficiency and strategy-proofness in exchange economies," *Games and Economic Behavior*, Vol. 86, pp. 26–39. [14]

——— (2018) "Probabilistic assignment: an extension approach," *Social Choice and Welfare*, Vol. 51, pp. 137–162. [14]

Cho, Wonki Jo and Battal Doğan (2016) "Equivalence of efficiency notions for ordinal assignment problems," *Economics Letters*, Vol. 146, pp. 8–12. [14]

Clarke, Edward H. (1971) "Multipart pricing of public goods," *Public Choice*, Vol. 11, No. 1, pp. 17–33, September. [12]

Cramton, Peter, Robert Gibbons, and Paul Klemperer (1987) "Dissolving a Partnership Efficiently," *Econometrica*, Vol. 55, No. 3, pp. 615–632, May. [4]

Gale, David and Lloyd S. Shapley (1962) "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, Vol. 69, pp. 9–15, Jan. [2]

Green, Jerry and Jean-Jacques Laffont (1977) "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, Vol. 45, pp. 417–438, March. [4], [12]

Groves, Theodore (1973) "Incentives in Teams," *Econometrica*, Vol. 41, No. 4, pp. 617–631, July. [13]

Gudmundsson, Jens (forthcoming) "Compromises and Rewards: stable and non-manipulable probabilistic matching," *International Journal of Game Theory*. [2]

Hausner, Melvin (1954) "Multidimensional utilities," in R.M. Thrall, C.H. Coombs, and R.L. Davis eds. *Decision Processes*: Wiley, pp. 167–180. [14]

Holmström, Bengt (1979) "Groves' Scheme on Restricted Domains," *Econometrica*, Vol. 47, No. 5, pp. 1137–1144, September. [4], [12]

Holmström, Bengt and Roger B. Myerson (1983) "Efficient and Durable Decision Rules with Incomplete Information," *Econometrica*, Vol. 51, No. 6, pp. 1799–1819, November. [2]

Hurwicz, Leonid (1972) "On informationally decentralized systems," in C. McGuire and R. Radner eds. *Decision and Organisation*: University of Minnesota Press, Chap. 14, pp. 297–336. [3]

Kasajima, Yoichi (2013) "Probabilistic assignment of indivisible goods with single-peaked preferences," *Social Choice and Welfare*, Vol. 41, No. 1, pp. 203–215, June. [3]

Manjunath, Vikram (2016) "Fractional matching markets," *Games and Economic Behavior*, Vol. 100, pp. 321–336. [1], [7], [10]

Momi, Takeshi (2017) "Efficient and strategy-proof allocation mechanisms in economies with many goods," *Theoretical Economics*, Vol. 12, No. 3, pp. 1267–1306, Sep. [3]

Myerson, Roger and Mark Satterthwaite (1983) "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, Vol. 29, No. 1, pp. 265–281, Apr. [4], [12]

Nesterov, Alexander S. (2017) "Fairness and efficiency in strategy-proof object allocation mechanisms," *Journal of Economic Theory*, Vol. 170, pp. 145–168. [3]

Roth, Alvin E., Uriel G. Rothblum, and John H. Vande Vate (1993) "Stable Matchings, Optimal Assignments, and Linear Programming," *Mathematics of Operations Research*, Vol. 18, No. 4, pp. 803–828, Nov. [1]

Saban, Daniela and Jay Sethuraman (2014) "A note on object allocation under lexicographic preferences," *Journal of Mathematical Economics*, Vol. 50, pp. 283–289. [14]

Schulman, Leonard J. and Vijay V. Vazirani (2012) "Allocation of Divisible Goods under Lexicographic Preferences," *CoRR*, Vol. abs/1206.4366. [14]

Schummer, James (1999) "Strategy-proofness versus efficiency for small domains of preferences over public goods," *Economic Theory*, Vol. 13, No. 3, pp. 709–722. [3]

——— (2000) "Eliciting Preferences to Assign Positions and Compensation," *Games and Economic Behavior*, Vol. 30, pp. 293–318. [4]

Serizawa, Shigehiro (2002) "Inefficiency of Strategy-Proof Rules for Pure Exchange Economies," *Journal of Economic Theory*, Vol. 106, No. 2, pp. 219–241, Oct. [3]

Sönmez, Tayfun (1999) "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica*, Vol. 67, No. 3, pp. 677–690, May. [2], [11], [13]

Thrall, Robert M. (1954) "Multidimensional utilities," in R.M. Thrall, C.H. Coombs, and R.L. Davis eds. *Decision Processes*: Wiley, pp. 181–186. [14]

Vickrey, William (1961) "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, Vol. 16, No. 1, pp. 8–37, Mar. [12]

von Neumann, John (1953) "A certain zero-sum two-person game equivalent to the optimal assignment problem," in H.W. Kuhn and A.W. Tucker eds. *Contributions to the theory of games, Vol. 2*: Princeton University Press. [5]

Yenmez, M. Bumin (2012) "Dissolving multi-partnerships efficiently," *Journal of Mathematical Economics*, Vol. 48, No. 2, pp. 77–82. [4]

——— (2013) "Incentive-Compatible Matching Mechanisms: Consistency with Various Stability Notions," *American Economic Journal: Microeconomics*, Vol. 5, No. 4, pp. 120–141, November. [4], [12]

——— (2015) "Incentive compatible market design with applications," *International Journal of Game Theory*, Vol. 44, No. 3, pp. 543–569, August. [4], [12]

Zhou, Lin (1990) "On a Conjecture by Gale about One-Sided Matching Problems," *Journal of Economic Theory*, Vol. 52, pp. 123–135. [3]